Chapter 2: What do groups look like?

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A road map for the Rubik's Cube

There are several solution techniques for the Rubik's Cube. If you do a quick Google search, you'll find several methods for solving the puzzle.

These methods describe a sequence of moves to apply relative to some starting position. In many situations, there may be a shorter sequence of moves that would get you to the solution.

Let's pretend for a moment that we were interested in writing a complete solutions manual for the Rubik's Cube. Let me be more specific about what I mean.

We'd like our solutions manual to have the following properties:

1. Given any scrambled configuration of the cube, there is a unique page in the manual corresponding to that configuration.
2. There is a method for looking up any particular configuration. (The details of how to do this are unimportant.)
3. Along with each configuration, a list of available moves is included. In each case, the page number for the outcome of each move is included and information about whether the corresponding move takes us closer to or farther from the solution.

Let's call our solutions manual the *Big Book*. See Figure 2.1 on page 13 for a picture of what a page in the *Big Book* might look like.

We can think of the *Big Book* as a road map for the Rubik's Cube. Each page says, "you are here" and "if you follow this road, you'll end up over there." In addition, you'll know whether "over there" is where you want to go or not.

Pros of the *Big Book*:

- We can solve any scrambled Rubik's Cube.
- In fact, given any configuration, every possible sequence of moves for solving the cube is listed in the book (long sequences and short sequences).
- The *Big Book* contains complete data on the moves in the Rubik's Cube universe and how they combine.

Cons of the *Big Book*:

- We just took all the fun out of the Rubik's Cube.
- If we had such a book, using it would be fairly cumbersome.
- We can't actually make such a book. Rubik's Cube has more than 4×10^{19} configurations. The paper required to write the book would cover the Earth many times over. The book would require over a billion terabytes of data to store electronically, and no computer in existence can store that much data.

Despite the *Big Book*'s apparent shortcomings, it made for a good thought experiment. The most important thing to get out of this discussion is that the *Big Book* is a map of a group.

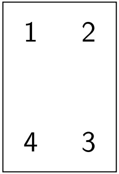
We shall not abandon the mapmaking ideas introduced by our discussion of the *Big Book* simply because the map is too large. We can use the same ideas to map out any group. In fact, we shall frequently do exactly that.

Let's try something simpler…

The Rectangle Puzzle

Here is the Rectangle Puzzle:

- Take a blank sheet of paper (our rectangle) and label as follows:



  This is the solved state of our puzzle.
- The idea of the game is to scramble the puzzle and then find a way to return the rectangle to its solved state.
- We are allowed two moves: horizontal flip and vertical flip, where "horizontal" and "vertical" refer to the motion of your hands, rather than any reference to an axis of reflection.

We'll spend some time in Chapter 3 discussing why these two moves and not some others are the ones that make sense for this game. However, it is worth pointing out that these two moves preserve the orientation of the rectangle. Are there any others that preserve its orientation?

Using only the two valid moves, scramble your rectangle. Any sequence of horizontal and vertical flips will do, but don't do any other types of moves.

Now, again using only our two valid moves, try to return your rectangle to the solved position.

Observations?

Question: do the moves of the Rectangle Puzzle form a group? How can we check?

For reference, here are the rules of a group:

Rule 1.5

There is a predefined list of actions that never changes.

Rule 1.6

Every action is reversible.

Rule 1.7

Every action is deterministic.

Rule 1.8

Any sequence of consecutive actions is also an action.

OK, let's see if we can make a road map for our newly found group.

Using our multiple copies of the rectangle, some colored yarn, and some sticky notes, let's see what we can come up with. (Someone remind me to take a picture when we are done.)

We've just created our first road map of a group! Observations? What sorts of things does the map tell us about the group?

We see that:

- the group has two generators: horizontal flip and vertical flip. Each generator is represented by the two different colors of yarn;
- the group has 4 actions: the "do nothing" action, horizontal flip, vertical flip, and 180° rotation ($r = h \circ v = v \circ h$);
- the map shows us how to get from any one configuration to any other (there may be more than one way to follow the yarn).

It is important to note that how we choose to layout our map is irrelevant. What is important is that the connections between the various states are preserved. However, we will attempt to construct our maps in a pleasing to the eye and symmetrical way.

The official name of the type of group road map that we have just created is Cayley diagram, named after the 19th century British mathematician Arthur Cayley.

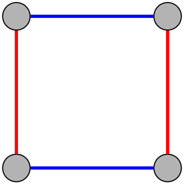
In general, a Cayley diagram consists of nodes that are connected by colored (or labeled) arrows, where

- an arrow of a particular color represents a specific generator;
- each action of the group is represented by a unique node (sometimes we will label nodes by the corresponding action);
- all necessary arrows are present (more on this later).

More on arrows:

- An arrow corresponding to the generator g from node A to node B means that node B is the result of applying the action g to node A.
- If the reverse of applying generator g is the same as g (this happens with horizontal and vertical flips), then we have a 2-way arrow. Our convention will be to drop the tips of the arrows on all 2-way arrows.

Here is one possible representation of the Cayley diagram for our Rectangle Puzzle:



The 2-Light Switch Group

Let's map out another group, which we'll call the 2-Light Switch Group. Here are the details:

- Consider two light switches side by side that both start in the off position.
- We are allowed 2 actions: flip L switch and flip R switch.

Do these actions generate a group?

In small groups, map out the 2-Light Switch Group using paper and yarn just like we did for the Rectangle Puzzle. (I suggest using U and D to denote "light switch up" and "light switch down", respectively.)

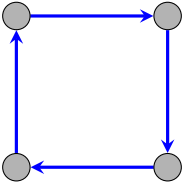
Now, draw the more abstract version of the Cayley diagram. What do you notice?

What we should notice is that the Cayley diagram for the Rectangle Puzzle and the Cayley diagram for the 2-Light Switch Group are essentially the same. The 4 rectangle configurations correspond to the 4 light switch configurations. Horizontal flip and vertical flip correspond to flip L switch and flip R switch.

Although these 2 groups are superficially different, the Cayley diagrams help us see that they have the same structure. (The fancy phrase for this phenomenon is that the “two groups are isomorphic”; more on this later.)

Any group with the same Cayley diagram as the Rectangle Puzzle and the 2-Light Switch Group is called the Klein 4-group, and is denoted by V_4 for *vierergruppe*, “four-group” in German. It is named after the mathematician Felix Christian Klein.

It is important to point out that the number of different types (i.e., colors) of arrows is important. For example, the following Cayley diagram does not represent V_4.



Warning: it is possible for two groups to have different looking Cayley diagrams yet really be the "same." (We'll talk more about what "same" means later.)

More Group Exercises

Let's explore a few more examples.

1. In groups of 2–3 (try to mix the groups up again), complete the following exercises (not collected):
    - Exercise 2.1 (see Bob)
    - Exercise 2.3 (see Bob)
    - Exercise 2.5
    - Exercise 2.8 (see Bob)
    - Exercise 2.10
    - Exercise 2.13 (see Bob)
2. I'd like each group to present their solution to one of the problems above.
3. Now, complete Exercise 2.18. I want each group to turn in a complete solution.

Potential quiz questions

Here are some potential questions that I may ask you on tomorrow's quiz at the beginning of class:

1. What do the arrows represent in a Cayley diagram?
2. What do the nodes represent in a Cayley diagram?
3. Draw 2 different Cayley diagrams and describe a specific set of actions (i.e., generators) that would yield the corresponding diagrams.