Chapter 3: Why study groups?

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In the previous 2 chapters, we introduced groups and explored a few basic examples. In this chapter, we shall discuss a few practical (yet not necessarily complicated) applications.

We will see applications of group theory in 3 areas:

1. science
2. art
3. mathematics

Our choice of examples is influenced by how well they illustrate the material rather than how useful they are.

Groups of symmetries

Intuitively, something is symmetrical when it looks the same from more than one point of view. Can you think of an object that you think exhibits symmetry? Have we already seen some?

How does symmetry relate to group? The examples of groups that we've seen so far deal with arrangements of similar things. In chapter 5, we shall uncover the following fact (we'll be more precise later):

> Every group can be viewed as a collection of ways to rearrange some set of things.

Groups relate to symmetry because an object's symmetries can be described using arrangements of the object's parts. The following definition provides a technique for finding a group that describes (or measures) a physical object's symmetry (in 3-D).

Definition 3.1

1. Identify all the parts of the object that are similar, and give each such part a different number.
2. Consider the actions that you could perform with your hands that may rearrange the numbered parts, yet leave the object taking up the same physical space it did originally. (This collection of actions forms a group.)
3. (Optional) If you want to visualize the group, explore and map it as we did in Chapter 2 with the rectangle, etc.

Comments

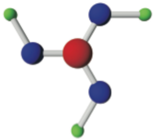
- We'll refer to the physical space that an object occupies as its footprint (this terminology does not appear on the text).
- Step 1 of Definition 3.1 numbers the object's parts so that we can track the manipulations permitted in Step 2. Each new state is a rearrangement of the object's similar parts and allows us to distinguish each of these rearrangements; otherwise we could not tell them apart.
- In this context, not *every* rearrangement of the similar parts is necessarily valid. We are only allowed actions that maintain the physical integrity of the object *and* preserve its footprint. For example, we can't rip two arms off a starfish and then glue them back on in different places.

Comments (continued)

- Step 2 requires us to find *all* of the actions that preserve the object's footprint and physical integrity; not just the generators.
- However, if we choose to complete Step 3 (construct Cayley diagram), we must make a choice concerning generators. As we mentioned in the previous chapter, different choices in generators may result in different Cayley diagrams.
- When selecting a set of generators, we would ideally like to select as small a set as possible. We can never choose too many generators, but we can choose too few. But having "extra" generators does nothing but clutter our Cayley diagram.

Because the shape of molecules impacts their behavior, chemists use group theory to classify their shapes. Let's take a look at an example.

The following figure (taken from page 28 of *Visual Group Theory*) depicts a molecule of Boric acid, $B(OH)_3$.

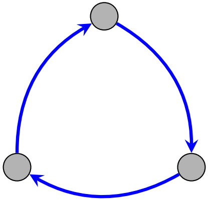


Follow the steps of Definition 3.1 to find the group that describes the symmetry of the molecule and draw a possible Cayley diagram.

What we should have discovered is that the group of symmetries of Boric acid has 3 actions requiring at least one generator. If we choose rotation clockwise 1/3 of a full turn as our generator, then the three actions are:

1. the do nothing action
2. rotation clockwise 1/3 of a full turn
3. rotation clockwise 2/3 of a full turn

The corresponding abstract Cayley diagram is as follows:



This is the cyclic group, C_3. (We'll discuss cyclic groups in Chapter 5.)

Group exercises

Let's explore a few more examples.

1. In groups of 2–3 (try to mix the groups up again), complete the following exercises (not collected):
   - Exercise 3.5
   - Exercise 3.6 (see Bob)
2. Let's discuss your solutions.
3. Now, complete Exercise 3.7. I want each group to turn in a complete solution.

Solids whose atoms arrange themselves in a regular, repeating pattern are called crystals. The study of crystals is called crystallography.

The wonderful picture in Figure 3.8 (page 30) shows the result of repeating indefinitely the crystal cube from Figure 3.7.

When chemists study such crystals they treat them as patterns that repeat without end. This allows a new manipulation that preserves the infinite footprint of the crystal and its physical integrity: translation.

In this case, the groups describing the symmetry of crystals are infinite. Why?

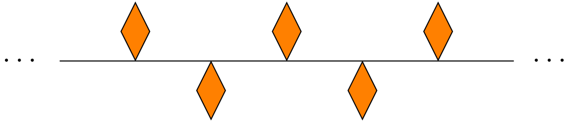
Frieze patterns

Crystals are patterns that repeat in 3 dimensions.

We will discuss simpler patterns that only repeat in one dimension, called frieze patterns. The groups that describe the symmetry of frieze patterns are called frieze groups.

Frieze patterns (or at least finite sections of them) occur throughout art and architecture.

Here is an example of a frieze pattern:



Because this frieze pattern extends infinitely far to the left and right, we are presented with a new type of manipulation that preserves the footprint and the physical integrity of the frieze. This new action is called a glide reflection and consist of a horizontal translation (by the appropriate amount) followed by a vertical flip.

Note that for this pattern, a vertical flip all by itself does not preserve the footprint, and so is not one of the actions of the group of symmetries.

Let's determine the group of symmetries of the frieze pattern on the previous slide and draw a possible Cayley diagram.

The group of symmetries of the frieze pattern on the previous slide turns out to be infinite, but we only needed two generators: horizontal flip and glide reflection. Figure 3.13 (page 33) depicts a possible Cayley diagram.

Comments

- The symmetry of any frieze pattern can be described by one of 7 different infinite groups. It turns out that some of the frieze groups are isomorphic (i.e., have the same structure) even though the visual appearance of the patterns may differ.
- The symmetry of 2-dimensional repeating patterns, called wallpaper patterns, has also been classified. There are 17 different wallpaper groups.

More group exercises

Time to do some more exploring.

1. In groups of 2–3 (try to mix the groups up again), complete the following exercises (not collected):
    - Exercise 3.11(a)
    - Exercise 3.11(b)
    - Exercise 3.11(d) (Bob may have something to say)
2. Let's discuss your solutions.

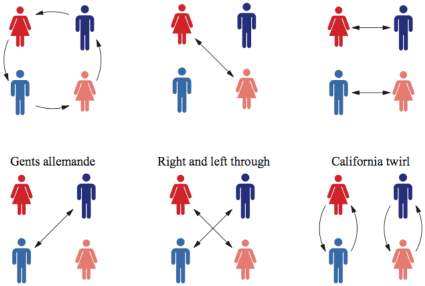
Contra dancing

In square dancing and contra dancing, the dancers follow a sequence of predefined steps called figures. Often dancers learn these steps by name and practice following a caller who orders them to perform specific figures in time with the music.

We'll assume that we have 2 couples standing in the shape of a square, so that individuals of the same sex are on opposite corners. To start, let's assume that one of the women is in the upper left hand corner of the square.

Dancing a figure rearranges the dancers. If they correctly obey the caller, every dance ends with the dancers back in their original positions in the square.

The following figure (taken from page 35 of *Visual Group Theory*) shows the effects of 6 example figures.



Do these 6 actions generate a group? The answer is yes (check the rules). It turns out, perhaps not surprisingly, that the group is isomorphic (i.e., same structure) as the group of symmetries of a square.

Even more group exercises

It's dance time!

1. In a large group, complete the following exercises (not collected):
   - Exercise 3.1
   - Exercise 3.13 (see Bob)
   - Exercise 3.14(a)
2. Let's discuss your solutions.
3. Now, in groups of 2–3, complete Exercise 3.15(a). I want each group to turn in a complete solution.

Potential quiz questions

Here are some potential questions that I may ask you on tomorrow's quiz at the beginning of class:

1. In order for an action to be a member of a group of symmetries for an object in 3-dimensions, what 2 important properties must this action have?
2. What is a glide reflection and to what kinds of objects can we apply them to?
3. Draw a Cayley diagram for a given molecule or frieze pattern.