Chapter 4: Algebra at last

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Notes

Recall that our informal definition of a group was a collection of actions that obeyed Rules 1.5–1.8. This is not the ordinary definition of a group.

In this chapter, we shall introduce the more standard (and more formal) definition of a group. We will also spend some time convincing ourselves that both definitions agree. (They should or we're in trouble!)

Along the way, we will also introduce another powerful visualization technique, called multiplication tables.

Notes

More on Cayley diagrams

Recall that the arrows in a Cayley diagram represent the generators of the group. In particular, all the arrows of a particular color correspond to the same unique generator.

Also, don't forget that our choice of generators influenced the resulting Cayley diagram.

By Rule 1.8, we know that any sequence of actions is an action. How are all the non-generator actions represented implicitly in a Cayley diagram?

The answer is that every action in the group is represented by a path through the diagram. Our immediate goal is to nail down exactly what this means.

Notes

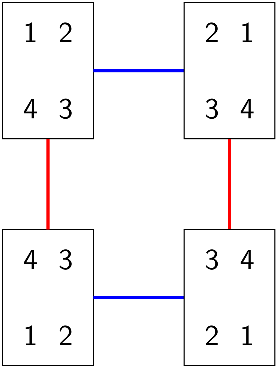
When we have been drawing Cayley diagrams, we have been doing one of two things with the nodes:

1. Labeling a node with a labeled configuration of thing we are acting on, so that the configuration at that node is the result of applying the generator corresponding to the arrow leading into that node.
2. Leaving the nodes unlabeled (I've referred to this as the abstract Cayley diagram).

Let's revisit an example we have already seen to help illustrate the point.

Notes

Consider the group of symmetries of a rectangle (alternatively, consider the 2-Lightswitch Group). As we've already discussed, this group has a total of 4 actions and we can use horizontal flip (h) and vertical flip (v) as generators. Here is a possible Cayley diagram, where we have labeled the nodes with configurations of of the rectangle and h is represented by the blue arrows and v is represented by the red arrows.



Notes

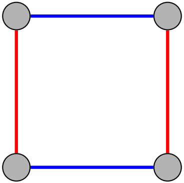
As we can see by looking at the Cayley diagram on the previous slide, following a sequence of arrows from one node to another shows us the result of applying the corresponding generators to the configurations we started with.

For example, if we start with the configuration in the upper left hand corner and then follow the blue edge (h) followed by the red edge (v), we end up at the configuration in the lower right hand corner. This sequence of actions is equivalent to a 180° rotation of the original configuration.

Do you see any other paths that represent this same action?

Notes

If we remove all reference to the specific configurations of the rectangle, we end up with an abstract Cayley diagram for V_4:



Now, while abstract Cayley diagrams are nice to look at, we definitely lose some information when we remove reference to the rectangle configurations.

What we'd like to do is strike a balance between these two representations. Since a group is a collection of actions (verbs), this will influence how we proceed.

Notes

Definition 4.1

The following steps transform a Cayley diagram into one that focuses on the group's actions.

(i) Choose a node as our initial reference point; label it e. (This will correspond to our "do nothing action.")

(ii) Relabel each remaining node in the diagram with a path that leads there from node e. (If there is more than one path, pick any one; shorter is better.)

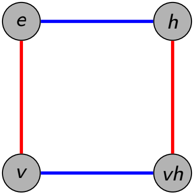
(iii) Distinguish arrows of the same type in some way (color them, label them, dashed vs. solid, etc.)

Our convention will be to label the nodes with sequences of generators, so that reading the sequence from left to right indicates the appropriate path. Warning: different authors often use the opposite convention.

Notes

The author calls the resulting diagram a diagram of actions. We will refer to these diagrams of actions as Cayley diagrams with the nodes labeled by actions (instead of configurations).

What do we get if we apply the steps to the abstract Cayley diagram for V_4? Here it is:



Note that we could also have labeled the node in the lower right hand corner as hv, as well. I'll emphasize this again later, but it is important to point out that this phenomenon (i.e., order of generators does not matter) does *not* always happen.

Notes

What do you think is a good way to represent the fact that doing a horizontal flip followed by a vertical flip results in the same action as doing a vertical flip followed by a horizontal flip? Yeah, that's right: $hv = vh$.

By the way, what if you forget which arrow corresponds to which generator? Just look to see what the label is on the node after following that arrow from e.

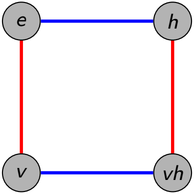
One of the really awesome things about Cayley diagrams with nodes labeled by actions is that we can use them as a sort of calculator.

What I mean by this is that if we want to know what a particular sequence (even really long ones!) is equal to, then we can just chase the sequence through the Cayley graph by starting at e.

Notes

Let's try one. In V_4, what is the action *hhhvhvvhv* equal to?

Here is the Cayley diagram for reference:



We see that $hhhvhvvhv = h$. A more condensed way to write this is $h^3vhv^2hv = h$. You might be wondering if we could have just written

$$h^5v^4 \stackrel{?}{=} h^3vhv^2hv = h.$$

Well, check it out! In this case, the answer is yes. Warning: not all groups have this property!!!

Notes

Group work

Let's explore a few more examples.

1. In groups of 2–3 (try to mix the groups up again), complete the following exercises (not collected):
   - Construct the Cayley diagram with nodes labeled by actions for the group of symmetries of an equilateral triangle (assume one tip of triangle is pointing up) using:
     (i) horizontal flip (h) and 120° rotation clockwise (r) as generators.
     (ii) horizontal flip (h) and the diagonal flip that keeps the lower left corner fixed (d).

     Any observations?
   - Exercise 4.17
   - Exercise 4.4
   - Exercise 4.5(a)
2. Let's discuss your solutions.

Notes

Since we can use a Cayley diagram with nodes labeled by actions as a calculator for figuring out what any length sequence of generators is equal to, we could create a table that shows how every pair of group actions combine. This type of table is called a (group) multiplication table.

This is best illustrated by diving in and doing an example. Using our Cayley diagram from earlier, let's see if we can complete the following multiplication table for V_4 using our generators h and v.

$*$	e	v	h	vh
e				
v				
h				
vh				

Notes

Comments

- The 1st column and 1st row repeat themselves. Why? Sometimes these will be omitted (*Group Explorer* does this).
- In each row and each column, each group action occurred exactly once. (This will always happen.)
- Multiplication tables can visually reveal patterns that may be difficult to see otherwise. To help make these patterns more obvious, we can color the cells of the multiplication table, where we assign a unique color to each action of the group. Figure 4.7 (page 47) has examples of a few such tables.

Notes

More group work

1. In groups of 2–3 (try to mix the groups up again), complete the following exercises (not collected):
   - Exercise 4.6(a)
   - Exercise 4.6(b)
2. Let's discuss your solutions.
3. Now, complete Exercise 4.19(a)(b)(c). I want each group to turn in a complete solution.

Notes

Moving towards the standard definition of a group

We have been calling the members that make up a group "actions" because our definition requires a group to be a collection of actions that satisfy our 4 rules. Since the standard definition of a group is not phrased in terms of actions, we will need more general terminology.

We will call the members of a group elements. In general, a group is a set of elements satisfying some set of properties.

We will also use standard set theory notation. For example, we will write things like

$$h \in V_4$$

to mean "the element h is an element of the group V_4."

Notes

Binary operations

Intuitively, an operation is a method for combining objects. For example, $+$, $-$, $\cdot$, and $\div$ are all examples of operations. In fact, these are all examples of binary operations because they combine two objects into a single object.

The combining of group elements is also a binary operation (like composition: do one action and then do another action to the result of the 1st one). We say that it is a binary operation *on* the group.

Binary operations on sets have the following special property.

If $*$ is a binary operation on a set S, then $s * t \in S$ for all $s, t \in S$.

The fancy way of saying this is that the set is closed under the binary operation.

Notes

Recall that Rule 1.8 says that any sequence of actions is an action. This ensures that the group was closed under the binary operation of combining actions.

Multiplication tables are nice because they depict the group's binary operation in full.

However, it is important to point out that not every table with symbols in it is going to be equal to the multiplication table for a group. Soon we will uncover a couple of features that distinguish those tables that depict groups from those that don't.

Notes

Associativity

Does anyone remember what it means for an operation to be associative? An operation is associative if parentheses are permitted anywhere, but required nowhere.

As examples, addition and multiplication of integers is associative. How we group a string together with parentheses has no impact on the outcome.

However, subtraction of integers is not associative. Here is an example:

$$3 - (2 - 4) \neq (3 - 2) - 4.$$

Notes

Is the operation of combining actions in a group associative? The answer is yes. We will not prove this fact, but rather illustrate it with an example.

Consider the group of symmetries for the equilateral triangle (called D_3 or S_3) with generators h and r from our group work earlier.

How do the following compare?

$$rhr, \quad (rh)r, \quad r(hr)$$

We see that even though we are associating differently, the end result is that the actions are applied left to right.

The moral of the story is that we do not ever need to use parentheses when working with groups, but sometimes we may use them to draw our attention to a particular chunk in a sequence.

Notes

Some more group work

In groups of 2–3, complete the following exercises (not collected):

- Exercise 4.14
- Exercise 4.10(a) (see Bob)

Notes

Inverses

Recall that Rule 1.6 requires every action to be reversible. Said another way, given any group element, you can find its opposite action, which we call its inverse.

If g represents some element (action) of a group, then we will use g^{-1} to denote the inverse of g.

Given any action of a group, what is the result of combining that action and its inverse (in either order)? Yep, we get the "do nothing action."

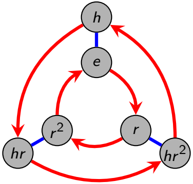
Our fancy word for the "do nothing action" is identity. We can really use whatever symbol we want to denote the identity of a group, but common choices are e, 1, 0, and N.

Using all of our new fancy notation, we can write expressions like

$$gg^{-1} = e \text{ and } g^{-1}g = e.$$

Notes

Let's explore these ideas a little more with one of our common examples. Recall that the Cayley diagram for the group of symmetries of the triangle (S_3 or D_3) is as follows.



Using the Cayley diagram, try to complete the following statements:

$$r^{-1} = \underline{\qquad\quad} \text{ because } r\underline{\qquad\quad} = e = \underline{\qquad\quad}r$$
$$h^{-1} = \underline{\qquad\quad} \text{ because } h\underline{\qquad\quad} = e = \underline{\qquad\quad}h$$
$$(hr)^{-1} = \underline{\qquad\quad} \text{ because } (hr)\underline{\qquad\quad} = e\underline{\qquad\quad}(hr)$$
$$(hr^2)^{-1} = \underline{\qquad\quad} \text{ because } (hr^2)\underline{\qquad\quad} = e = \underline{\qquad\quad}(hr^2).$$

Notes

Some more group exercises

1. In groups of 2–3, complete the following exercises (not collected):
   - Exercise 4.10(b)
   - Exercise 4.11(a)
   - Exercise 4.26(a)
2. Let's discuss your solutions.
3. Now, in groups of 2–3, complete Exercise 4.27(a)(b). I want each group to turn in a complete solution for both parts.

Notes

Classical definition of a group

We are now ready to state the standard definition of a group.

Definition 4.2

A set G is a group if the following criteria are satisfied.

1. There is a binary operation $*$ on G.
2. $*$ is associative.
3. There is an identity element $e \in G$. That is, $e * g = g = g * e$.
4. Every element $g \in G$ has an inverse, g^{-1}, satisfying $g * g^{-1} = e = g^{-1} * g$.

Notes

Do our two competing definitions agree? That is, if Definition 1.9 says something is a group, will Definition 4.2 agree? Or vice versa?

Our discussion leading up to Definition 4.2 provides an informal argument for why the answer to the first question must be yes. We will answer the second question in the next chapter.

Regardless of whether the definitions agree (which they do), we always have $e^{-1} = e$. That is, the reverse of doing nothing is doing nothing.

Even though we haven't officially shown that the two definitions agree, we shall begin viewing groups from these two different paradigms:

- group as a collection of actions
- group as a set with a binary operation

Notes

Even more group exercises

In groups of 2–3, complete Exercise 4.32. I want each group to turn in a complete solution.

Notes

Potential quiz questions

Here are some potential questions that I may ask you on tomorrow's quiz at the beginning of class:

1. What is a binary operation?
2. What is our second definition of a group?
3. Determine whether a given multiplication table represents a group.
4. State at least two properties that *all* groups share.
5. Find expression for the inverse of a group element.
6. Solve a specified group equation for a particular group.

Notes