Diagram algebras and applications to Kazhdan–Lusztig theory

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Definition

A Coxeter system (W, S) consists of a group W (called a Coxeter group) generated by a set S of involutions with presentation

$$W = \langle S : s^2 = 1, (st)^{m(s,t)} = 1 \rangle,$$

where $m(s,t) \geq 2$ for $s \neq t$.

Comment

Since s and t are involutions, the relation $(st)^{m(s,t)} = 1$ can be rewritten as

$$m(s,t) = 2 \implies st = ts \quad \text{(short braid relations)}$$

$$\left.\begin{array}{l} m(s,t) = 3 \implies sts = tst \\ m(s,t) = 4 \implies stst = tsts \\ \quad \vdots \end{array}\right\} \quad \text{long braid relations}$$

Definition

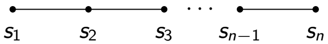
We can encode (W, S) with a unique Coxeter graph Γ having:

- vertex set S;
- edges $\{s, t\}$ labeled $m(s, t)$ whenever $m(s, t) \geq 3$ (if $m(s, t) = 3$, we omit label).

Comments

- If s and t are not connected in Γ, then s and t commute.
- W is irreducible if Γ is connected.
- Given Γ, we can uniquely reconstruct the corresponding (W, S). In this case, we may denote the group and corresponding generating set by $W(\Gamma)$ and $S(\Gamma)$, respectively.

Coxeter groups of type A_n ($n \geq 1$) are defined by:



Then $W(A_n)$ is generated by $S(A_n) = \{s_1, s_2, \cdots, s_n\}$ and is subject to defining relations

1. $s_i^2 = 1$ for all i,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$,
3. $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$.

$W(A_n)$ is isomorphic to the symmetric group, S_{n+1}, under the correspondence

$$s_i \mapsto (i \ i+1),$$

where $(i \ i+1)$ is the adjacent transposition exchanging i and $i+1$.

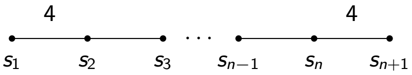
Coxeter groups of type B_n ($n \geq 2$) are defined by:

In this case, $W(B_n)$ is generated by $S(B_n) = \{s_1, s_2, \cdots, s_n\}$ and is subject to defining relations

1. $s_i^2 = 1$ for all i,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$,
3. $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$ and $1 < i, j \leq n$,
4. $s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$.

$W(B_n)$ is a finite group of order $2^n n!$ (wreath product of $\mathbb{Z}_2$ and the symmetric group).

Coxeter groups of type $\widetilde{C}_n$ ($n \geq 2$), pronounced "affine C_n," are defined by:



Here, we see that $W(\widetilde{C}_n)$ is generated by $S(\widetilde{C}_n) = \{s_1, \cdots, s_{n+1}\}$ and is subject to defining relations

1. $s_i^2 = 1$ for all i,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$,
3. $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$ and $1 < i, j < n + 1$,
4. $s_i s_j s_i s_j = s_j s_i s_j s_i$ if $\{i, j\} = \{1, 2\}$ or $\{n, n+1\}$.

$W(\widetilde{C}_n)$ is an infinite group.

Comment

We can obtain $W(A_n)$ and $W(B_n)$ from $W(\widetilde{C}_n)$ by removing the appropriate generators and the corresponding relations. In fact, we can obtain $W(B_n)$ in two ways.

Definition

A word $s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ is called an expression for $w \in W$ if it is equal to w when considered as a group element.

If m is minimal, it is a reduced expression, and the length of w is $\ell(w) := m$.

Given $w \in W$, if we wish to emphasize a fixed, possibly reduced, expression for w, we represent it as

$$\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_m}.$$

Theorem (Matsumoto)

Any two reduced expressions for $w \in W$ differ by a sequence of braid relations.

Example

Let $w \in W(B_3)$ with expression $\overline{w} = s_1 s_2 s_1 s_2 s_3 s_1$. Since $s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$, $s_1 s_3 = s_3 s_1$, and $s_1^2 = 1$ in $W(B_3)$, we see that

$$s_1 s_2 s_1 s_2 s_3 s_1 = s_2 s_1 s_2 s_1 s_3 s_1 = s_2 s_1 s_2 s_1 s_1 s_3 = s_2 s_1 s_2 s_3.$$

This shows that $\overline{w}$ is not reduced. However, it is true (but not immediately obvious) that $s_2 s_1 s_2 s_3$ is a reduced expression for w, so that $l(w) = 4$.

Comment

Applying a commutation or a long braid does not change the length of an expression. Only applying relations of the form $s^2 = 1$ can reduce length.

Definition

We say that $w \in W$ is fully commutative (FC) if any two reduced expressions for w can be transformed into each other via iterated commutations. The set of FC elements of W is denoted by $\mathrm{FC}(W)$.

Theorem (Stembridge)

$w \in W$ is FC iff no reduced expression for w contains a long braid.

Comments

The FC elements of $W(\widetilde{C}_n)$ are precisely those that avoid the following consecutive subexpressions:

1. $s_i s_j s_i$ for $|i-j| = 1$ and $1 < i, j < n+1$,
2. $s_i s_j s_i s_j$ for $\{i, j\} = \{1, 2\}$ or $\{n, n+1\}$.

It follows from work of Stembridge that $W(\widetilde{C}_n)$ contains an infinite number of FC elements. There are examples of infinite Coxeter groups that contain a finite number of FC elements (e.g., type E_n for $n \geq 9$).

Example

Let $w \in W(\widetilde{C}_3)$ have reduced expression $\overline{w} = s_1 s_3 s_2 s_1 s_2$. Since s_1 and s_3 commute, we can write

$$w = s_1 s_3 s_2 s_1 s_2 = s_3 s_1 s_2 s_1 s_2.$$

This shows that w has a reduced expression containing $s_1 s_2 s_1 s_2$ as a consecutive subexpression, which implies that w is not FC.

Now, let $w' \in W(\widetilde{C}_3)$ have reduced expression $\overline{w}' = s_1 s_2 s_1 s_3 s_2$. Then we will never be able to rewrite w' to produce one of the illegal consecutive subexpressions since the only relation we can apply is

$$s_1 s_3 \to s_3 s_1$$

which does not provide an opportunity to apply any additional relations. So, w' is FC.

Let (W, S) be a Coxeter system with graph Γ. The associated Hecke algebra is an algebra with a basis indexed by the elements of W and relations that deform the relations of W by a parameter q. If we set q to 1, we recover the group algebra of W. More specifically:

Definition

The associative $\mathbb{Z}[q, q^{-1}]$-algebra $\mathcal{H}_q(\Gamma)$ is the free module on the set $\{T_w : w \in W\}$ that satisfies

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } l(sw) > l(w), \\ qT_{sw} + (q-1)T_w, & \text{otherwise.} \end{cases}$$

We extend the scalars to $\mathcal{A} := \mathbb{Z}[v, v^{-1}]$, where $v^2 = q$:

$$\mathcal{H}(\Gamma) := \mathcal{A} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{H}_q(\Gamma).$$

We call $\mathcal{H}(\Gamma)$ the Hecke algebra associated to W.

Comments

- If $\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_m}$ is a reduced expression for $w \in W$, then
$$T_w = T_{s_{x_1}} T_{s_{x_2}} \cdots T_{s_{x_m}}.$$
- $\mathcal{A}$ has a ring automorphism $\bar{\ }$ sending $v \mapsto v^{-1}$. This "extends" to a ring automorphism $\bar{\ } : \mathcal{H}(\Gamma) \to \mathcal{H}(\Gamma)$ satisfying
$$\overline{T_w} = (T_{w^{-1}})^{-1}.$$
$\bar{\ }$ is like inverse the revenge!
- Define $\widetilde{T_w} = v^{-l(w)} T_w$. Then $\{\widetilde{T_w} : w \in W\}$ is an $\mathcal{A}$-basis for $\mathcal{H}(\Gamma)$.
- We define $\mathcal{L}$ to be the free $\mathbb{Z}[v^{-1}]$-module on the set $\widetilde{T_w}$. There exists a natural map $\pi : \mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L}$.

Theorem (Kazhdan, Lusztig)

There is a unique basis $\{C'_w : w \in W\}$ *for* $\mathcal{H}(\Gamma)$ *satisfying:*

1. $\overline{C'_w} = C'_w$
2. $C'_w \in \mathcal{L}$ *and* $\pi(C'_w) = \pi\left(\widetilde{T_w}\right)$.

This basis has important and subtle properties. (Called the canonical basis).

Definition

The Kazhdan–Lusztig polynomials occur as follows. If

$$C'_w = \sum_{y \leq w} P^*_{y,w} \widetilde{T_y},$$

where $\leq$ is the Bruhat order on the Coxeter group W, then

$$P_{y,w} := v^{l(w)-l(y)} P^*_{y,w}.$$

Comments

- $P_{y,w} = 0$ unless $y \leq w$ (Bruhat order).
- $P_{w,w} = 1$ for all $w \in W$.
- $P_{y,w} \in \mathbb{Z}[q]$. In fact, $\mathbb{Z}_{\geq 0}[q]$ . . . deep!
- If $P_{y,w} \neq 0$, then $\deg(P_{y,w}) \leq \frac{1}{2}(l(w) - l(y) - 1)$
- We write $\mu(y, w) \in \mathbb{Z}$ for the coefficient of $q^{1/2(l(w)-l(y)-1)}$ in $P_{y,w}$. Clearly, $\mu(y, w) = 0$ unless both $y < w$ and $l(w)$ and $l(y)$ have different parity.
- There is a (terrifying looking!) recursive formula

$$P_{x,w} = q^{1-c} P_{sx,v} + q^c P_{x,v} - \sum_{z \prec v, sz < z} \mu(z, w) q^{1/2(l(w)-l(z)-1)} P_{x,z},$$

where $sw = v < w$ and $c = \begin{cases} 0, & \text{if } x < sx \\ 1, & \text{otherwise.} \end{cases}$

Comments

- K–L polynomials have applications to the representation theory of semisimple algebraic groups, Verma modules, algebraic geometry and topology of Schubert varieties, etc.
- There is natural basis indexed by the elements of W for $\mathcal{H}$: $\{T_w\}$.
- There is this another really nice basis that we like better: $\{C'_w\}$.
- The K–L polynomials essentially occur as the entries in the change of basis matrix from one basis to the other.
- The μ-values occur as the coefficients on the highest degree term in the corresponding K–L polynomial.
- Unfortunately, computing the polynomials efficiently quickly becomes difficult, even in finite groups of moderate size.
- Computing the μ-values is helpful, but not known to be any easier.

0–1 Conjecture

In type A_n, $\mu(y, w)$ is always 0 or 1.

Theorem (McLarnan, Warrington)

0–1 Conjecture fails in type A_9 and up.

Comment

Conjecture does hold for some special classes of elements.

Theorem

In type A_n, if y is FC, then $\mu(y, w)$ is always 0 or 1.

Current Research

There are quite a few people (like myself) trying to find non-recursive ways to compute K–L polynomials and/or μ-values for various Coxeter groups.

Definition

Let (W, S) be a Coxeter system with graph Γ. Define $J(\Gamma)$ be the two-sided ideal of $\mathcal{H}(\Gamma)$ generated by

$$\sum_{w \in \langle s, s' \rangle} T_w,$$

where (s, s') runs over all pairs of of elements of $S(\Gamma)$ with $3 \leq m(s, s') < \infty$, and $\langle s, s' \rangle$ is the (parabolic) subgroup generated by s and s'. We define the (generalized) Temperley–Lieb algebra, $\mathrm{TL}(\Gamma)$, to be the quotient $\mathcal{A}$-algebra $\mathcal{H}(\Gamma)/J(\Gamma)$.

Theorem (Graham)

Let t_w denote the image of T_w in the quotient. Then the set $\{t_w : w \in \mathrm{FC}(W)\}$ is an $\mathcal{A}$-basis for $\mathrm{TL}(\Gamma)$.

Comment

Green and Losonczy have show that $\mathrm{TL}(\Gamma)$ admits a canonical basis, $\{c_w : w \in \mathrm{FC}(W)\}$. This basis is analogous to the K–L basis for $\mathcal{H}(\Gamma)$ and in many situations, c_w is known to be the image of C'_w in the quotient (conjectured to always be the case).

Definition

For each $s_i \in S$, define $b_i = v^{-1}t_{s_i} + v^{-1}t_e$. If $w \in \mathrm{FC}(W)$ has reduced expression $\overline{w} = s_{x_1} \cdots s_{x_m}$, define

$$b_w = b_{x_1} \cdots b_{x_m}.$$

Theorem (Graham)

The set $\{b_w : w \in \mathrm{FC}(W)\}$ *forms an* $\mathcal{A}$*-basis for* $\mathrm{TL}(\Gamma)$. *(Called the monomial basis.)*

Theorem (Graham)

$\mathrm{TL}(\widetilde{C}_n)$ *is generated (as unital algebra) by* $b_1, b_2, \ldots, b_{n+1}$ *with defining relations*

1. $b_i^2 = \delta b_i$ *for all* i, *where* $\delta = v + v^{-1}$
2. $b_i b_j = b_j b_i$ *if* $|i-j| > 1$,
3. $b_i b_j b_i = b_i$ *if* $|i-j| = 1$ *and* $1 < i, j < n+1$,
4. $b_i b_j b_i b_j = 2 b_i b_j$ *if* $\{i,j\} = \{1,2\}$ *or* $\{n, n+1\}$.

$\mathrm{TL}(A_n)$ *and* $\mathrm{TL}(B_n)$ *are generated by* $b_2, \ldots, b_n$ *and* $b_1, b_2, \ldots, b_n$, *respectively, together with the corresponding relations.*

History

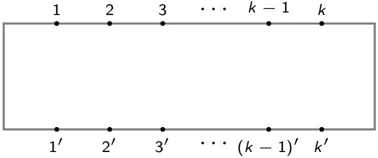
- The algebra $\mathrm{TL}(A_n)$ was invented in 1971 by Temperley and Lieb and first arose in the context of integrable Potts models in statistical mechanics.
- As well as having applications in physics, $\mathrm{TL}(A_n)$ appears in the framework of knot theory, braid groups, Coxeter groups and their corresponding Hecke algebras, and subfactors of von Neumann algebras.
- Penrose/Kauffman use diagram algebra to model $\mathrm{TL}(A_n)$ in 1971.
- In 1987, Vaughan Jones recognized that $\mathrm{TL}(A_n)$ is isomorphic to a particular quotient of the Hecke Algebra of type A_n (the symmetric group).
- Since 1987, there have been various generalizations of Temperley– Lieb type quotients and related diagram algebras.

Motivation

- One motivation behind studying $\mathrm{TL}(\Gamma)$ is that it provides a gateway to understanding the K–L theory of the associated Hecke algebra.
- Loosely speaking, $\mathrm{TL}(\Gamma)$ retains some of the relevant structure of $\mathcal{H}(\Gamma)$, yet is small enough that the computation of the μ-values of the K–L polynomials is often simpler.

Definition

A standard k-box is a rectangle with $2k$ nodes, labeled as follows:



A concrete pseudo k-diagram consists of a finite number of disjoint curves (planar), called edges, embedded in and disjoint from the standard k-box such that

1. edges may be closed (isotopic to circles), but not if their endpoints coincide with the nodes of the box;
2. the nodes of the box are the endpoints of curves, which meet the box transversely.

Definition (continued)

Two concrete pseudo k-diagrams are (isotopically) equivalent if one concrete diagram can be obtained from the other by isotopically deforming the edges such that any intermediate diagram is also a concrete pseudo k-diagram.

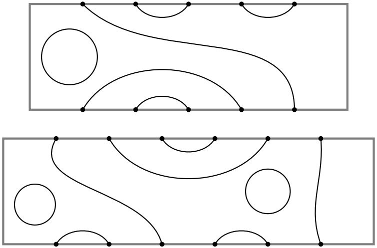
A pseudo k-diagram (or an ordinary Temperley-Lieb pseudo diagram) is defined to be an equivalence class of equivalent concrete pseudo k-diagrams.

An edge joining i in the N-face to j' in the S-face is called a propagating edge. All other edges are called non-propagating.

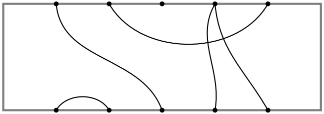
Let's look at some examples.

Example

Here are two examples of concrete pseudo diagrams.



Here is an example that is not a concrete pseudo diagram.



Definition

The (type A) Temperley–Lieb diagram algebra, denoted by $\mathbb{D}\mathrm{TL}(A_n)$, is the free $\mathbb{Z}[\delta]$-module with basis consisting of the pseudo $(n+1)$-diagrams having no loops.

We define multiplication by defining multiplication in the case where d and d' are basis elements (i.e., loop-free pseudo diagrams), and then extend bilinearly.

To calculate the product dd' identify the "S-face" of d with the "N-face" of d' and then multiply by a factor of δ for each resulting loop and then discard the loop.

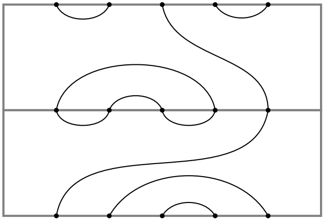
$\mathbb{D}\mathrm{TL}(A_n)$ is an associative $\mathbb{Z}[\delta]$-algebra having the loop-free pseudo $(n+1)$-diagrams as a basis.

Comment

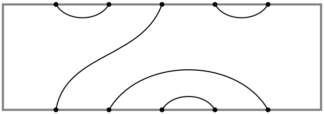
A typical element of $\mathrm{TL}(A_n)$ looks like a linear combination of loop-free pseudo $(n+1)$-diagrams, where the coefficients are polynomials in δ.

Example

Multiplication of two concrete pseudo 5-diagrams.

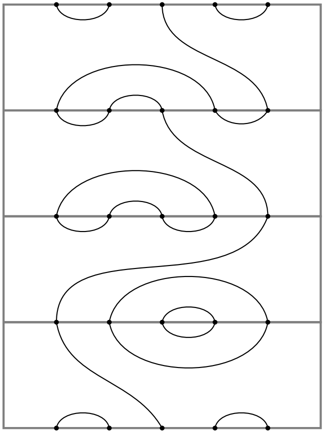


$= \delta$

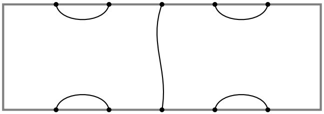


Example

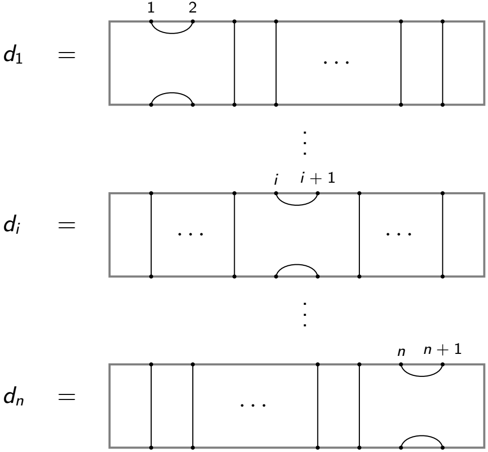
And here's another example.



$= \delta^3$



Now, we define the set of simple pseudo $(n+1)$-diagrams, which turn out to form a generating set for $\mathbb{D}\mathrm{TL}(A_n)$.



Theorem

The $\mathbb{Z}[\delta]$-algebra homomorphism $\theta : \mathsf{TL}(A_n) \to \mathbb{D}TL(A_n)$ determined by

$$\theta(b_i) = d_i$$

is an algebra isomorphism. Moreover, the loop-free pseudo $(n+1)$-diagrams are in bijection with the monomial basis elements of $\mathsf{TL}(A_n)$.

Theorem (R.M. Green)

If $y, w \in W(A_n)$ with both FC, then $\mu(y, w)$ can be computed (non-recursively) as follows.

1. *Draw diagrams for d_y and $d_{w^{-1}}$.*
2. *Multiply d_y times $d_{w^{-1}}$. Do not replace any closed loops with δ.*
3. *Connect point i in N-face to point i' in S-face (without intersections).*

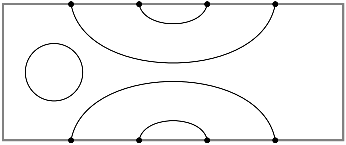
If this forms n closed loops, then $\mu(y, w) = 1$, and otherwise, $\mu(y, w) = 0$.

Example

Let $\overline{y} = s_2$ and $\overline{w} = s_2 s_1 s_3 s_2$ be reduced expressions for y and w, respectively, in $W(A_3)$. Note that both y and w are FC. We see that $w^{-1} = s_2 s_3 s_1 s_2$. Then

$$d_y d_{w^{-1}} = d_2 d_2 d_3 d_1 d_2,$$

which yields the following pseudo diagram:



If we "wrap up" this diagram, we see that there are 3 loops. Therefore, by the previous theorem, $\mu(y, w) = 1$.

Comments

- What we are really doing when we "wrap up" $d_y d_{w^{-1}}$ is defining a trace function on a quotient of the Hecke algebra.
- This trace function is a generalized Jones trace and satisfies the Markov property.
- When this type of trace function is known to exist, we can use it to compute $\mu(y, w)$ for $y, w \in \mathrm{FC}(W)$.
- At this point, only when we have a diagrammatic representation of $\mathrm{TL}(\Gamma)$ have we been able to define the necessary trace so that we can non-recursively compute μ-values.
- Current state of affairs: we can do this when Γ is of type A, B, D, H, E, or $\widetilde{A}$. (See papers by R.M. Green.)
- Coming soon: type $\widetilde{C}$!
- Elusive: type F.