# A diagrammatic representation of an affine $C$ Temperley–Lieb algebra

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Thesis Defense[1]
May 29, 2008

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[1] Revised on 1.16.09: a few minor corrections made.

# Coxeter Groups

**Definition**

A Coxeter group is a group $W$ with together with a distinguished set of generating involutions $S$ subject to relations of the form

$$(st)^{m(s,t)} = 1,$$

where $m(s,s) = 1$ and $m(s,t) = m(t,s)$.

We call the pair $(W, S)$ a Coxeter system.

### Remark

If $s, t \in S$, then the relation

$$(st)^{m(s,t)} = 1$$

can be rewritten as

$$\underbrace{stst\cdots}_{m(s,t)} = \underbrace{tsts\cdots}_{m(s,t)} \tag{1}$$

since $s$ and $t$ are involutions. In particular, if $m(s,t) = 2$, then

$$st = ts.$$

That is, $s$ and $t$ commute when $m(s,t) = 2$. If $m(s,t) \geq 3$, then we refer to (1) as a long braid relation.

### Definition

Given a Coxeter system $(W, S)$, the associated Coxeter graph is the graph $X$ with:

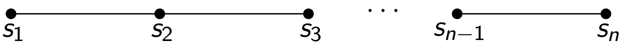
1. vertex set equal to $S$;
2. edges connecting $s_i$ and $s_j$ labeled $m(s_i, s_j)$ for all pairs $i, j$ with $m(s_i, s_j) > 2$. If $m(s_i, s_j) = 3$ it is customary to leave the corresponding edge unlabeled.

### Remark

Given a Coxeter graph $X$, we can reconstruct the corresponding Coxeter system $(W, S)$.

### Example

The Coxeter graph of type $A_n$ ($n \geq 1$) is as follows.




Then $W(A_n)$ is generated by $S(A_n) = \{s_1, s_2, \cdots, s_n\}$ and is subject to defining relations

1. $s_i^2 = 1$ for all $i$,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$,
3. $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$.

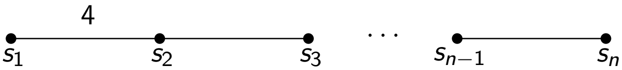
$W(A_n)$ is isomorphic to the symmetric group, $S_{n+1}$, under the correspondence

$$s_i \mapsto (i \;\; i+1),$$

where $(i \;\; i+1)$ is the adjacent transposition exchanging $i$ and $i+1$.

## Example

The Coxeter graph of type $B_n$ ($n \geq 2$) is as follows.




In this case, $W(B_n)$ is generated by $S(B_n) = \{s_1, s_2, \cdots, s_n\}$ and is subject to defining relations

1. $s_i^2 = 1$ for all $i$,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$,
3. $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$ and $1 < i, j \leq n$,
4. $s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$.

$W(B_n)$ is a finite group of order $2^n n!$.

## Example

The Coxeter graph of type $\widetilde{C}_n$ ($n \geq 2$), pronounced "affine $C_n$," is as follows.


Here, we see that $W(\widetilde{C}_n)$ is generated by $S(\widetilde{C}_n) = \{s_1, \cdots, s_{n+1}\}$ and is subject to defining relations

1. $s_i^2 = 1$ for all $i$,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$,
3. $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$ and $1 < i, j < n + 1$,
4. $s_i s_j s_i s_j = s_j s_i s_j s_i$ if $\{i, j\} = \{1, 2\}$ or $\{n, n + 1\}$.

$W(\widetilde{C}_n)$ is an infinite group.

### Definition

Let $X$ be an arbitrary Coxeter graph. An expression is any product of generators from $S(X)$. The length $l(w)$ of an element $w \in W(X)$ is the minimum number of generators appearing in any expression for the element $w$. Such a minimum length expression is called a reduced expression.

Each element $w \in W(X)$ can have several different reduced expressions that represent it. Given $w \in W(X)$, if we wish to emphasize a fixed, possibly reduced, expression for $w$, we represent it as

$$\overline{w} = s_{i_1} \cdots s_{i_k},$$

where each $s_{i_j} \in S(X)$.

### Example

Let $w \in W(B_3)$ with expression $\overline{w} = s_1 s_2 s_1 s_2 s_3 s_1$. Since $s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$, $s_1 s_3 = s_3 s_1$, and $s_1^2 = 1$ in $W(B_3)$, we see that

$$s_1 s_2 s_1 s_2 s_3 s_1 = s_2 s_1 s_2 s_1 s_3 s_1 = s_2 s_1 s_2 s_1 s_1 s_3 = s_2 s_1 s_2 s_3.$$

This shows that $\overline{w}$ is not reduced. However, it is true (but not immediately obvious) that $s_2 s_1 s_2 s_3$ is a reduced expression for $w$, so that $l(w) = 4$.

## Definition

Let $w \in W(X)$. We write

$$\mathcal{L}(w) = \{s \in S(X) : l(sw) < l(w)\}$$

and

$$\mathcal{R}(w) = \{s \in S(X) : l(ws) < l(w)\}.$$

The set $\mathcal{L}(w)$ (respectively, $\mathcal{R}(w)$) is called the left (respectively, right) descent set of $w$.

It turns out that $s \in \mathcal{L}(w)$ (respectively, $\mathcal{R}(w)$) iff $w$ has a reduced expression beginning (respectively, ending) with $s$.

### Example

Let $w \in W(B_4)$ have reduced expression $\overline{w} = s_1 s_3 s_2 s_1$. Since $s_1$ and $s_3$ commute, but $s_2$ commutes with neither $s_1$ nor $s_3$, it follows (from Matsumoto's Theorem) that

$$\mathcal{L}(w) = \{s_1, s_3\}$$

and

$$\mathcal{R}(w) = \{s_1\}.$$

## Remark

It is known to be true that we can obtain $W(B_n)$ from $W(\widetilde{C}_n)$ by removing the generator $s_{n+1}$ and the corresponding relations. We also obtain a Coxeter group of type $B$ if we remove the generator $s_1$ and the corresponding relations.

To distinguish these two cases, we let $W(B_n)$ denote the subgroup of $W(\widetilde{C}_n)$ generated by

$$S(\widetilde{C}_n) \setminus \{s_{n+1}\} = \{s_1, s_2, \ldots, s_n\}$$

and we let $W(B'_n)$ denote the subgroup of $W(\widetilde{C}_n)$ generated by

$$S(\widetilde{C}_n) \setminus \{s_1\} = \{s_2, s_3, \ldots, s_{n+1}\}.$$

# Fully commutative elements of Coxeter groups

**Definition**

We say that $w \in W(X)$ is fully commutative if any two reduced expressions for $w$ may be transformed into each other by iterated commutations.

**Theorem (Stembridge)**

*$w \in W(X)$ is fully commutative iff no reduced expression for $w$ contains a long braid as a consecutive subexpression.*

**Remark**

The fully commutative elements of $W(\widetilde{C}_n)$ are precisely those such that all reduced expressions avoid consecutive subexpressions of the following types:

1. $s_i s_j s_i$ for $|i - j| = 1$ and $1 < i, j < n + 1$,
2. $s_i s_j s_i s_j$ for $\{i, j\} = \{1, 2\}$ or $\{n, n + 1\}$.

## Remark (continued)

It follows from work of Stembridge that $W(\widetilde{C}_n)$ contains an infinite number of fully commutative elements. There are examples of infinite Coxeter groups that contain a finite number of fully commutative elements.

We denote the set of fully commutative elements of $W(X)$ by $W_c(X)$.

## Example

Let $w \in W(\widetilde{C}_3)$ have reduced expression $\overline{w} = s_1 s_3 s_2 s_1 s_2$. Since $s_1$ and $s_3$ commute, we can write

$$w = s_1 s_3 s_2 s_1 s_2 = s_3 s_1 s_2 s_1 s_2.$$

This shows that $w$ has a reduced expression containing $s_1 s_2 s_1 s_2$ as a consecutive subexpression, which implies that $w$ is not fully commutative.

Example (continued)

Now, let $w' \in W(\widetilde{C}_3)$ have reduced expression $\overline{w}' = s_1 s_2 s_1 s_3 s_2$. Then we will never be able to rewrite $w'$ to produce one of the illegal consecutive subexpressions since the only relation we can apply is

$$s_1 s_3 \longrightarrow s_3 s_1$$

and this does not provide an opportunity to apply any additional relations. So, $w'$ is fully commutative.

# Weak star reductions

We now introduce the concept of a weak star reduction, which generalizes ordinary star reductions and is similar to Fan's notion of cancelable.

Definition

Let $X$ be a Coxeter graph and let $w \in W_c(X)$. Suppose that $s \in \mathcal{L}(w)$. Then $w$ is left weak star reducible by $s$ with respect to $t$ to $sw$ if

1. $t \in \mathcal{L}(sw)$;
2. $m(s,t) \geq 3$;
3. $tw \notin W_c(X)$.

We analogously define right weak star reducible.

### Definition (continued)

If $w$ is not left or right weak star reducible by any $s \in S$, then we say that $w$ is weak star irreducible, or simply irreducible.

### Example

Let $w, w' \in W_c(\widetilde{C}_n)$ (for $n \geq 4$) have reduced expressions $\overline{w} = s_1 s_2 s_5$ and $\overline{w}' = s_1 s_2 s_1 s_5$, respectively. We see that $w'$ is left (and right) weak star reducible by $s_1$ with respect to $s_2$, and so $w'$ is not irreducible. However, $w$ is irreducible.

# Classification of the weak star irreducible elements

**Theorem (Fan, Ernst)**

*$w \in W_c(B_n)$ is irreducible iff $w$ is equal to one of the elements on the following list.*

(i) $w_p$;

(ii) $s_1 s_2 w_p$, *where* $s_1, s_2, s_3 \notin \mathrm{supp}(w_p)$;

(iii) $s_2 s_1 w_p$, *where* $s_1, s_2, s_3 \notin \mathrm{supp}(w_p)$;

*where in each case $w_p$ is equal to a product of commuting generators.*

*We have an analogous statement for $W_c(B_n')$, where $s_1$ and $s_2$ are replaced with $s_{n+1}$ and $s_n$, respectively.*

## Remark

The previous theorem verifies Fan's unproved claim about the type $B$ cancelable elements. (The proof is nontrivial.)

Before stating the classification of the type $\widetilde{C}_n$ irreducible elements, we need some notation.

## Definition

Define the following elements of $W(\widetilde{C}_n)$.

1. If $i < j$, let

$$\overline{z}_{i,j} = s_i s_{i+1} \cdots s_{j-1} s_j$$

   and

$$\overline{z}_{j,i} = s_j s_{j-1} \cdots s_{i-1} s_i.$$

2. If $1 < i \le n+1$ and $1 \le j < n+1$, let

$$\overline{z}_{i,j}^{\overbrace{LRL\cdots RL}^{k \text{ alternating factors}}} = \overline{z}_{i,2}(\overline{z}_{1,n}\overline{z}_{n+1,2})^{\frac{k-1}{2}}\overline{z}_{1,j},$$

where $k$ must be odd, so that $k-1$ is even.

3. If $1 < i \le n+1$ and $1 < j \le n+1$, let

$$\overline{z}_{i,j}^{\overbrace{LRL\cdots RLR}^{k \text{ alternating factors}}} = \overline{z}_{i,2}(\overline{z}_{1,n}\overline{z}_{n+1,2})^{\frac{k-2}{2}}\overline{z}_{1,n}\overline{z}_{n+1,j},$$

where $k$ must be even, so that $k-1$ is odd.

4. If $1 \leq i < n+1$ and $1 \leq j < n+1$, let

$$\overline{z}_{i,j}^{\overbrace{RLR\cdots LRL}^{k \text{ alternating factors}}} = \overline{z}_{i,n}(\overline{z}_{n+1,2}\overline{z}_{1,n})^{\frac{k-2}{2}}\overline{z}_{n+1,2}\overline{z}_{1,j},$$

where $k$ must be even, so that $k-1$ is odd.

5. If $1 \leq i < n+1$ and $1 < j \leq n+1$, let

$$\overline{z}_{i,j}^{\overbrace{RLR\cdots LR}^{k \text{ alternating factors}}} = \overline{z}_{i,n}(\overline{z}_{n+1,2}\overline{z}_{1,n})^{\frac{k-1}{2}}\overline{z}_{n+1,j},$$

where $k$ must be odd, so that $k-1$ is even.

We will refer to these elements as type I.

## Example

Consider $W(\widetilde{C}_4)$. Then

$$\overline{z}^{R}_{1,1} = s_1 s_2 s_3 s_4 s_5 s_4 s_3 s_2 s_1.$$

Also, we have

$$\overline{z}^{LRL}_{2,3} = s_2 s_1 s_2 s_3 s_4 s_5 s_4 s_3 s_2 s_1 s_2 s_3.$$

## Remark

It will be helpful for us to define $l = \lceil \frac{n-1}{2} \rceil$. Then regardless of whether $n$ is odd or even, $2l$ (respectively, $2l+1$) will always be the largest even (respectively, odd) number amongst $\{1, 2, \ldots, n, n+1\}$.

Definition

Define $\mathcal{O} = \{1, 3, \ldots, 2l-1, 2l+1\}$ and $\mathcal{E} = \{2, 4, \ldots, 2l-2, 2l\}$. Then define

$$\overline{x}_{\mathcal{O}} = s_1 s_3 \cdots s_{2l-1} s_{2l+1},$$

and

$$\overline{x}_{\mathcal{E}} = s_2 s_4 \cdots s_{2l-2} s_{2l}.$$

We will refer to finite alternating products of $\overline{x}_{\mathcal{O}}$ and $\overline{x}_{\mathcal{E}}$ as type II elements.

Example

Let $w \in W_c(\widetilde{C}_4)$ have reduced expression $s_1 s_3 s_5 s_2 s_4 s_1 s_3 s_5$. Then $w$ is of type II.

### Theorem (Ernst)

*An element $w \in W_c(\widetilde{C}_n)$ is irreducible iff $w$ is equal to one of the elements on the following list.*

(i) *$uv$, where $u$ is a type $B$ irreducible element and $v$ is a type $B'$ irreducible element such that $\mathrm{supp}(u) \cap \mathrm{supp}(v) = \emptyset$;*

(ii) *$\overline{z}^{R*R}_{1,1}$, $\overline{z}^{L*L}_{n+1,n+1}$, $\overline{z}^{L*R}_{n+1,1}$, and $\overline{z}^{R*L}_{1,n+1}$ (these are the type I elements with left and right descent sets equal to either $s_1$ or $s_{n+1}$);*

(iii) *any type II element.*

# Hecke algebras

**Definition**

Let $X$ be an arbitrary Coxeter graph. We define the Hecke algebra of type $X$, denoted by $\mathcal{H}_q(X)$, to be the $\mathbb{Z}[q, q^{-1}]$-algebra with basis consisting of (invertible) elements $T_w$, for all $w \in W(X)$, satisfying

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ qT_{sw} + (q-1)T_w & \text{if } l(sw) < l(w) \end{cases}$$

where $s \in S(X)$ and $w \in W(X)$.

It is convenient to extend the scalars of $\mathcal{H}_q(X)$ to produce an $\mathcal{A}$-algebra, $\mathcal{H}(X) = \mathcal{A} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{H}_q(X)$, where $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ and $v^2 = q$.

# Temperley–Lieb algebras

**Definition**

Let $J(X)$ be the two-sided ideal of $\mathcal{H}(X)$ generated by

$$\sum_{w \in \langle s, s' \rangle} T_w,$$

where $(s, s')$ runs over all pairs of of elements of $S(X)$ with $3 \leq m(s, s') < \infty$, and $\langle s, s' \rangle$ is the subgroup generated by $s$ and $s'$.

Following Graham, we define the (generalized) Temperley–Lieb algebra, $\mathrm{TL}(X)$, to be the quotient $\mathcal{A}$-algebra $\mathcal{H}(X)/J(X)$.

## Theorem (Graham)

*Let $t_w$ denote the image of $T_w$ in the quotient. Then the set $\{t_w : w \in W_c(X)\}$ is an $\mathcal{A}$-basis for $\mathrm{TL}(X)$.*

For our purposes, it will be more useful to work with a different basis.

## Definition

For each $s_i \in S(X)$, define $b_i = v^{-1} t_{s_i} + v^{-1} t_e$, where $e$ is the identity in $W(X)$. If $w \in W_c(X)$ has reduced expression $\overline{w} = s_{i_1} \cdots s_{i_r}$, then we define

$$b_w = b_{i_1} \cdots b_{i_r}.$$

(It turns out that this definition is independent of choice of reduced expression.)

The following two theorems are implicit in J. Graham's thesis.

**Theorem**

*The set $\{b_w : w \in W_c(X)\}$ forms an $\mathcal{A}$-basis for $\mathrm{TL}(X)$. This basis is referred to as the monomial basis.*

**Theorem**

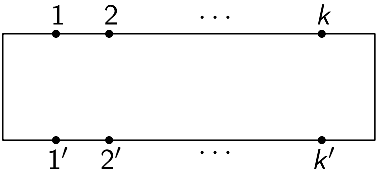
*The infinite dimensional $\mathcal{A}$-algebra $\mathrm{TL}(\widetilde{C}_n)$ is generated as a unital algebra by $b_1, b_2, \ldots, b_{n+1}$ with defining relations*

1. $b_i^2 = \delta b_i$ *for all* $i$*, where* $\delta = v + v^{-1}$
2. $b_i b_j = b_j b_i$ *if* $|i - j| > 1$,
3. $b_i b_j b_i = b_i$ *if* $|i - j| = 1$ *and* $1 < i, j < n + 1$,
4. $b_i b_j b_i b_j = 2 b_i b_j$ *if* $\{i, j\} = \{1, 2\}$ *or* $\{n, n + 1\}$.

# Ordinary Temperley–Lieb diagrams

### Definition

Let $k$ be a nonnegative integer. The standard $k$-box is a rectangle with $2k$ marks points, called nodes labeled as follows.




A concrete pseudo $k$-diagram consists of a finite number of disjoint curves (planar), called edges, embedded in and disjoint from the standard $k$-box such that

1. edges may be closed (isotopic to circles), but not if their endpoints coincide with the nodes of the box;
2. the nodes of the box are the endpoints of curves, which meet the box transversely.

## Definition (continued)

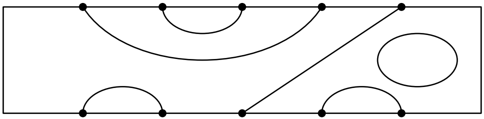
An edge joining $i$ in the north face to $j'$ in the south face is called a propagating edge. All other edges are called non-propagating.

Two concrete pseudo $k$-diagrams are (isotopically) equivalent if one concrete diagram can be obtained from the other by isotopically deforming the edges such that any intermediate diagram is also a concrete pseudo $k$-diagram.

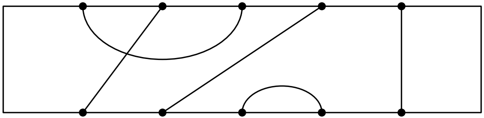
A pseudo $k$-diagram (or an ordinary Temperley-Lieb pseudo diagram) is defined to be an equivalence class of equivalent concrete pseudo $k$-diagrams.

## Example

Here is an example of a concrete pseudo 5-diagram.



Here an example of a drawing that is not a concrete pseudo 5-diagram.



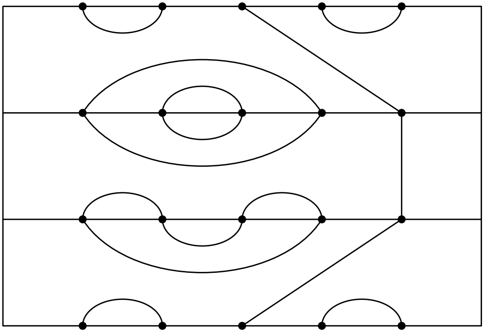
Definition
The (ordinary) Temperley–Lieb diagram algebra, denoted by $\mathbb{D}\mathrm{TL}(A_n)$, is the free $\mathbb{Z}[\delta]$-module with basis consisting of the pseudo $(n+1)$-diagrams having no loops.

We define multiplication by defining multiplication in the case where $d$ and $d'$ are basis elements (i.e., loop-free pseudo diagrams), and then extend bilinearly. To calculate the product $dd'$ identify the "south face" of $d$ with the "north face" of $d'$ and then multiply by a factor of $\delta$ for each resulting loop and then discard the loop.

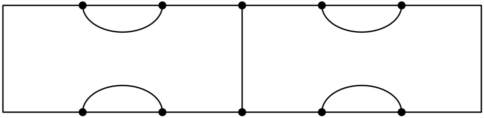
$\mathbb{D}\mathrm{TL}(A_n)$ is an associative $\mathbb{Z}[\delta]$-algebra having the loop-free pseudo $(n+1)$-diagrams as a basis.

## Example

Here is an example of multiplication of three basis diagrams of $\mathbb{D}\mathrm{TL}(A_4)$.



$= \delta^3$



Theorem

*As* $\mathbb{Z}[\delta]$*-algebras,* $\mathrm{TL}(A_n) \cong \mathbb{D}\mathrm{TL}(A_n)$*. Moreover, the loop-free pseudo* $(n+1)$*-diagrams are in bijection with the monomial basis elements of* $\mathrm{TL}(A_n)$*.*

We now describe a particular diagram algebra, where the diagrams are allowed to carry decorations.

# Decorated diagrams

Let $\mathcal{V} = \{\bullet, \blacktriangle, \circ, \triangle\}$. This will be our decoration set, where each element is called a decoration. The first two decorations are called closed and the other two are called open. Any finite sequence of decorations is called a block.

Fix $n \geq 2$. Let $d$ be a fixed concrete pseudo $(n+2)$-diagram and let $e$ be an edge of $d$. We may adorn $e$ with a finite (possibly empty) sequence of blocks of decorations such that adjacency of blocks and decorations is preserved as we travel along $e$. Each decoration on $e$ has an associated $y$-coordinate in the plane, which we will call its vertical position.

We require the following:

- If $d$ has no non-propagating edges (i.e., all edges are "vertical"), then we require $d$ to be undecorated.
- It is possible to deform all decorated edges of $d$ so as to take open decorations to the left and closed decorations to the right simultaneously.
- If $e$ is non-propagating, then we allow adjacent blocks on $e$ to be conjoined to form larger blocks.
- If $d$ has more than 1 non-propagating edge in north face and $e$ is propagating, then we allow adjacent blocks on $e$ to be conjoined to form larger blocks.

- If $d$ has exactly one non-propagating edge in north face and $e$ is propagating, then we allow $e$ to be decorated subject to the following constraints:
    1. All decorations occurring on propagating edges must have vertical position lower (respectively, higher) than the vertical positions of decorations occurring on the (unique) non-propagating edge in the north face (respectively, south face) of $d$.
    2. If $\mathbf{b}$ is block of decorations occurring on $e$, then no other decorations occurring on any other propagating edges may have vertical position in the range of vertical positions that $\mathbf{b}$ occupies.
    3. If $\mathbf{b}_i$ and $\mathbf{b}_{i+1}$ are two adjacent blocks occurring on $e$, then they may be conjoined to form a larger block only if the previous requirement is not violated.

### Definition

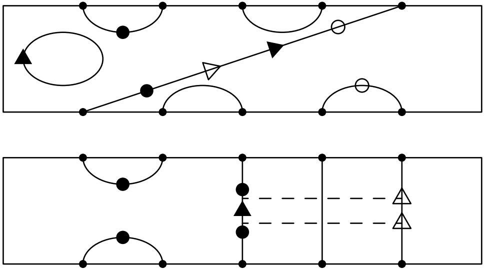
A concrete LR-decorated pseudo $(n+2)$-diagram is any $\mathcal{V}$-decorated concrete diagram that satisfies the conditions given above.

We define two concrete pseudo LR-decorated $(n+2)$-diagrams to be $\mathcal{V}$-equivalent if we can isotopically deform one diagram into the other such that any intermediate diagram is also a concrete pseudo LR-decorated $(n+2)$-diagram.

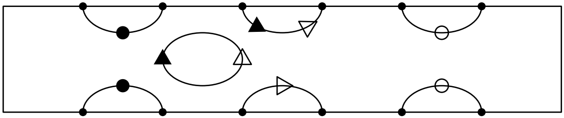
An LR-decorated pseudo $(n+2)$-diagram is defined to be an equivalence class of $\mathcal{V}$-equivalent concrete LR-decorated pseudo $(n+2)$-diagrams.

## Example

Here are two examples of LR-decorated pseudo 5-diagrams.



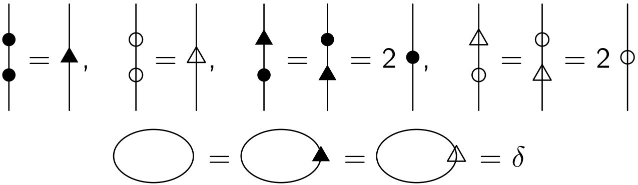
And here is an example of an LR-decorated pseudo 6-diagram.



## Definition

We define $\widehat{\mathcal{P}}^{LR}_{n+2}(\mathcal{V})$ to be the free $\mathbb{Z}[\delta]$-module with basis consisting of the set of LR-decorated diagrams having blocks that do not contain any adjacent decorations of the same type (open and closed) and does not have any of the loops listed below.

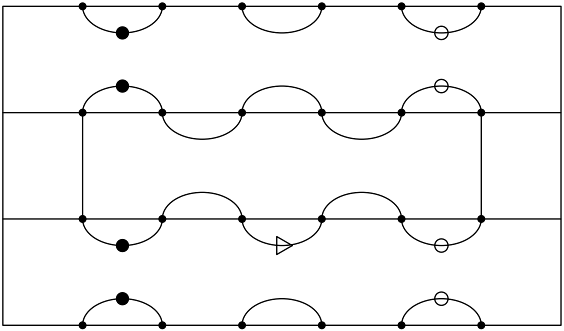
We define multiplication by defining multiplication in the case where $d$ and $d'$ are basis elements, and then extend bilinearly. To calculate the product $dd'$, concatenate $d$ and $d'$. While maintaining $\mathcal{V}$-equivalence, conjoin adjacent blocks subject to the following relations:



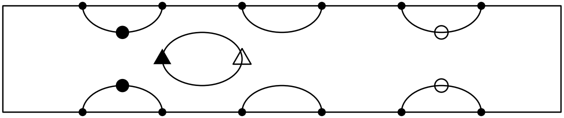
**Theorem (Ernst)**

*The multiplication defined above turns $\widehat{\mathcal{P}}^{LR}_{n+2}(\mathcal{V})$ into a well-defined associative $\mathbb{Z}[\delta]$-algebra. A basis for $\widehat{\mathcal{P}}^{LR}_{n+2}(\mathcal{V})$ consists of the LR-decorated diagrams having blocks that do not contain any adjacent decorations of the same type (open and closed) and there are no loops that can be replaced with $\delta$.*

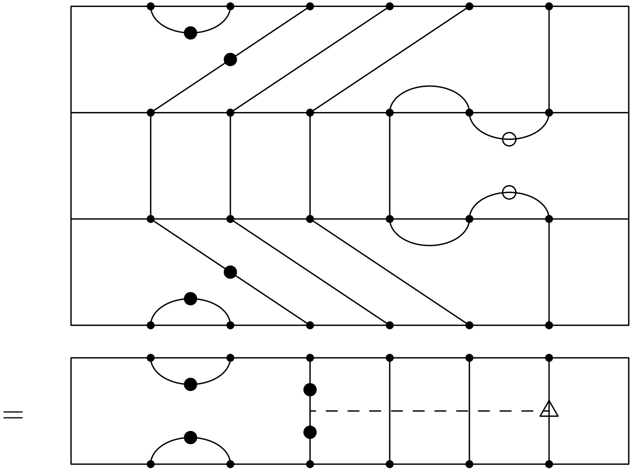
# Example



$= 2$

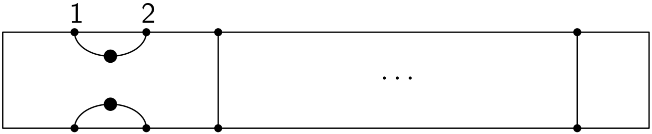


# Example

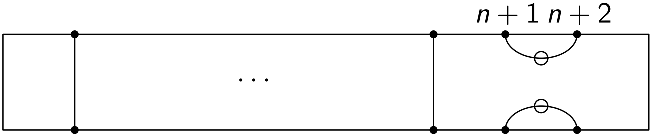


We define the simple diagrams $d_1, d_2, \ldots, d_{n+1}$ as follows.

$d_1 =$

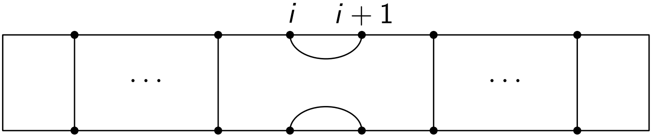



$d_{n+1} =$




For $1 < i < n+1$:

$d_i =$



Note that the simple diagrams lie in $\widehat{\mathcal{P}}^{LR}_{n+2}(\mathcal{V})$.

### Definition

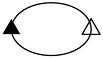
We define $\mathbb{D}_n$ to be the $\mathbb{Z}[\delta]$-subalgebra of $\widehat{\mathcal{P}}^{LR}_{n+2}(\mathcal{V})$ generated by the simple diagrams.

Now, we describe a basis for $\mathbb{D}_n$.

### Definition

Let $d$ be an LR-decorated diagram. Then we say that $d$ is admissible, if the following axioms are satisfied.

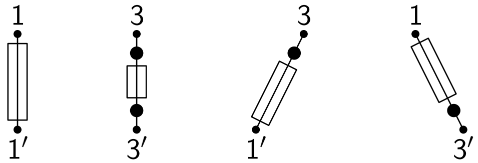
(C1) The only loops that may appear are equivalent to the following.



(C2) If $d$ has no propagating edges (which can only happen if $n$ is even), then the edges joining nodes 1 and $1'$ (respectively, nodes $n+2$ and $(n+2)'$) must be decorated with a $\bullet$ (respectively, $\circ$). Furthermore, these are the only $\bullet$ (respectively, $\circ$) decorations that may occur on $d$ and must be the leftmost (respectively, rightmost) decorations on their respective edges.

(C3) If $d$ has exactly one propagating edge $e$ (which can only happen if $n$ is odd), then $e$ may be decorated by an alternating sequence of $\blacktriangle$ and $\vartriangle$ decorations. If $e$ is connected to node 1 (respectively, $1'$), then the highest (respectively, lowest) decoration occurring on $e$ must be a $\bullet$. Similarly, if $e$ is connected to node $n+2$ (respectively, $(n+2)'$), then the highest (respectively, lowest) decoration occurring on $e$ must be a $\circ$. Furthermore, if there is a non-propagating edge connected to 1 or $1'$ (respectively, $n+2$ or $(n+2)'$) it must be decorated only by a single $\bullet$ (respectively, $\circ$). Finally, no other $\bullet$ or $\circ$ decorations appear on $d$.

(C5) If $d$ has exactly one non-propagating edge in the north face, then the leftmost propagating edge is equal to one of the following, where the rectangle represents a sequence of blocks (possibly empty), where each block is a single ▲.




Also, the occurrences of the • decorations occurring on the propagating edge are the highest or lowest decorations occurring on any propagating edge. We have an analogous requirement for the rightmost propagating edge, where the closed decorations are replaced with open decorations. Furthermore, if there is a non-propagating edge connected to 1 or $1'$ (respectively, $n+2$ or $(n+2)'$) it must be decorated only by a single • (respectively, ∘). Finally, no other • or ∘ decorations appear on $d$.

(C4) Assume that $d$ has more than one non-propagating edge and more than one propagating edge. If there is a propagating edge joining 1 to $1'$ (respectively, $n+2$ to $(n+2)'$), then it is decorated by a single $\blacktriangle$ (respectively, $\vartriangle$). Otherwise, an edge joining only one of 1 or $1'$ (respectively, $n+2$ or $(n+2)'$) is decorated by a single $\bullet$ (respectively, $\circ$) and there are no other $\bullet$ or $\circ$ decorations appearing on $d$.

### Theorem (Ernst)

*The admissible diagrams form a basis for $\mathbb{D}_n$.*

# Main result

**Theorem (Ernst)**

*Let $\theta : \mathrm{TL}(\widetilde{C}_n) \to \mathbb{D}_n$ be the function determined by*

$$\theta(b_i) = d_i.$$

*Then $\theta$ is an algebra isomorphism of $\mathrm{TL}(\widetilde{C}_n)$ and $\mathbb{D}_n$. Moreover, the admissible diagrams are in bijection with the monomial basis elements of $\mathrm{TL}(\widetilde{C}_n)$.*

The hard part is proving that $\theta$ is injective. The classification of the irreducible elements provides the groundwork for inductive arguments that are used to prove faithfulness.

# Who cares?

The monomial basis for $\mathrm{TL}(\widetilde{C}_n)$ is not the basis that we are really interested in. We sweat blood and tears proving that we have a faithful representation of the monomial basis, so that we can perform a change of basis of the diagram algebra. A topic of future research is to show that this new diagram algebra basis coincides with the so-called "canonical basis" of $\mathrm{TL}(\widetilde{C}_n)$. Using this new faithful representation, we define a trace on $\mathcal{H}(\widetilde{C}_n)$ that would be very difficult to define without the diagrammatic representation in hand. Using this trace, we will be able to non-recursively compute leading coefficients of certain Kazhdan–Lusztig polynomials, which are notoriously difficult to compute.

# Special thanks

I'd like to thank my family (Jen, Tristan, and Kaden) and my advisor (Richard Green) for being so patient with me.

I'd also like to thank the other members of my committee for reading my thesis and coming today, especially Dr. Nat Thiem and Dr. Chanyoung Lee Shader for traveling so far to be here.