Diagram Calculus for the Temperley-Lieb Algebra

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Definition

Let n be a positive integer. The *Temperley-Lieb Algebra*, $\mathrm{TL}_n(\delta)$, with parameter δ is defined to be the associative, unital algebra over the ring $\mathbb{Z}[\delta]$ generated by elements $e_1, e_2, \ldots, e_{n-1}$ subject only to the relations

$$e_i^2 = \delta e_i, \text{ for all } i$$
$$e_i e_j = e_j e_i, \text{ for } |i-j| \geq 2$$
$$e_i e_j e_i = e_i, \text{ for } |i-j| = 1$$

Theorem

$\mathrm{TL}_n(\delta)$ is a finite dimensional associative algebra over $\mathbb{Z}[\delta]$. A basis may be described in terms of "reduced words" in the algebra generators e_i. The rank of $\mathrm{TL}_n(\delta)$ is the nth Catalan number:

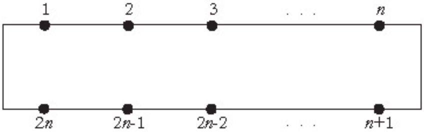
$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}.$$

Some Remarks:

- $\mathrm{TL}_n(\delta)$ was invented in 1971 by Temperley and Lieb.
- First arose in the context of integrable Potts models in statistical mechanics.
- As well as having applications in physics, $\mathrm{TL}_n(\delta)$ appears in the framework of knot theory, braid groups, Coxeter groups and their corresponding Hecke algebras, and subfactors of von Neumann algebras.
- Penrose/Kauffman use diagram algebra to model $\mathrm{TL}_n(\delta)$ in 1971.
- In 1987, Vaughan Jones recognized that $\mathrm{TL}_n(\delta)$ is isomorphic to a particular quotient of the Hecke Algebra of type A_{n-1} (the symmetric group, S_n).

Definition

A *standard n-box* is a rectangle with $2n$ nodes, labeled as follows:

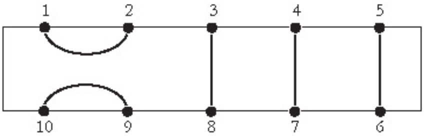


An *n-diagram* is a graph drawn on the nodes of a standard n-box such that

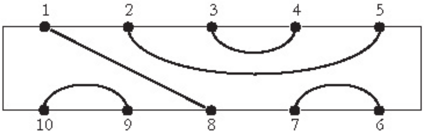
- Every node is connected to exactly one other node by a single edge.
- All edges must be drawn inside the n-box.
- The graph can be drawn so that no edges cross.

Example

Here is an example of a 5-diagram.

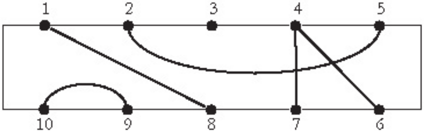


Here is another.



Example

Here is an example that is *not* a diagram.



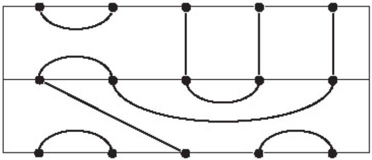
Definition

The associative *diagram algebra*, $\mathcal{D}_n(\delta)$, is the free $\mathbb{Z}[\delta]$-module having the set of n-diagrams as a basis with multiplication defined as follows.

If d and d' are n-diagrams, then dd' is obtained by identifying the "south face" of d with the "north face" of d', and then replacing any closed loops with a factor of δ.

Example

Multiplication of two 5-diagrams.



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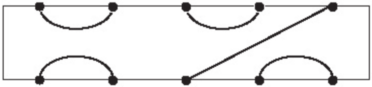
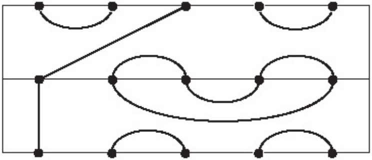
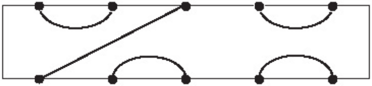


Example

Here's another example.

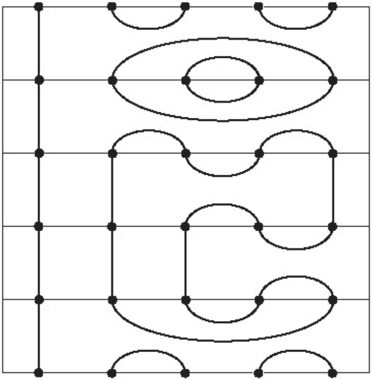


$= \delta$

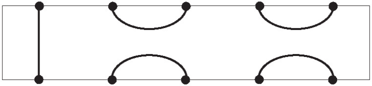


Example

And here's one more.



$= \ \delta^3$

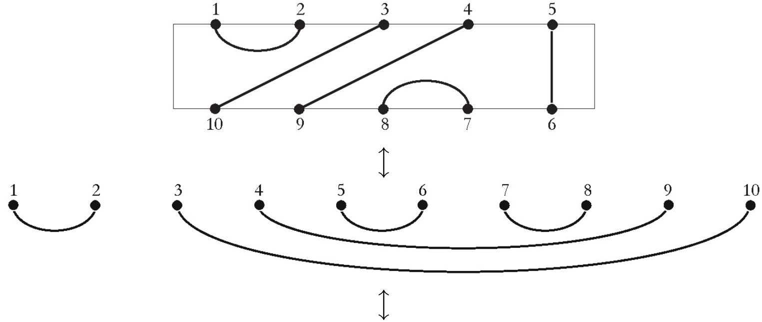


Theorem

The rank of the diagram algebra $\mathcal{D}_n(\delta)$ is C_n.

Proof.

The number of sequences of n pairs of well-formed parentheses is C_n. There is a one-to-one correspondence between n-diagrams and sequences of n pairs of well-formed parentheses.



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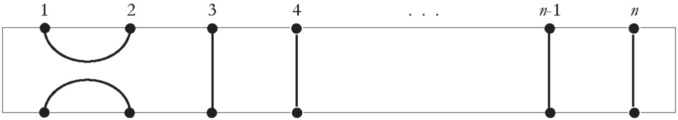
□

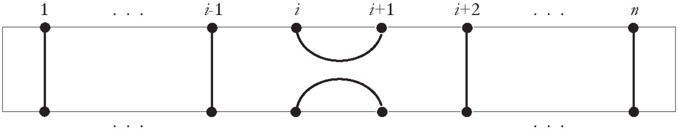
Now, we define a few “simple” n-diagrams. Let

$d_1 =$



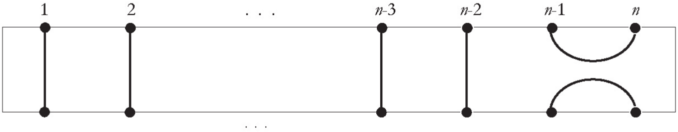
$\vdots$

$d_i =$



$\vdots$

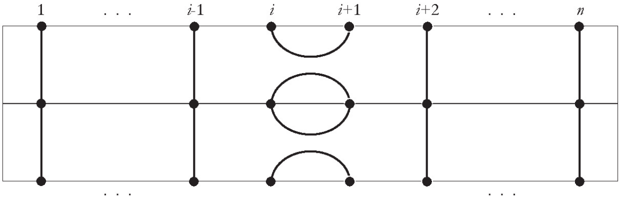
$d_{n-1} =$

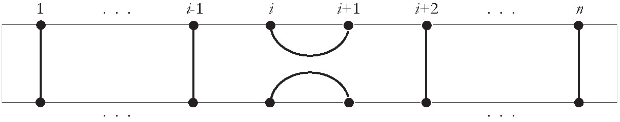


Claim 1: The diagrams $d_1, d_2, \ldots, d_{n-1}$ generate $\mathcal{D}_n(\delta)$.

Claim 2: The generators $d_1, d_2, \ldots, d_{n-1}$ satisfy the relations of $\mathrm{TL}_n(\delta)$.

For all i, we have

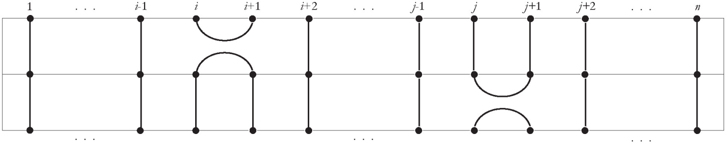
$d_i^2 =$ 

$= \delta$ 

$= \delta d_i$

For $|i - j| \geq 2$, we have

$d_i d_j =$



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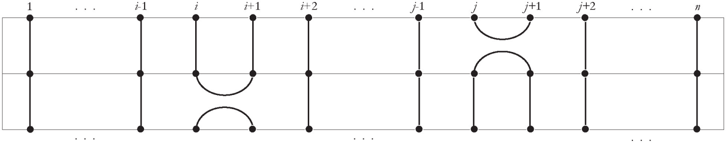
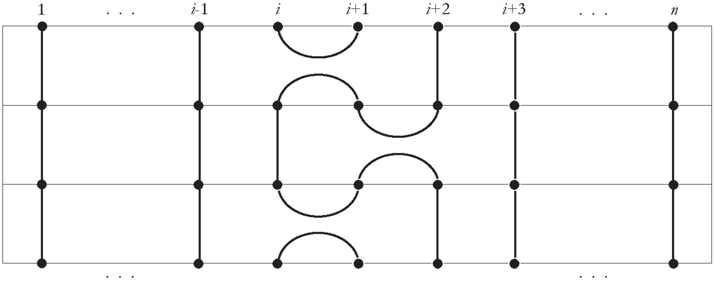


$= d_j d_i$

For $|i-j|=1$ (here, $j=i+1$; $j=i-1$ being similar), we have

$d_i d_j d_i =$



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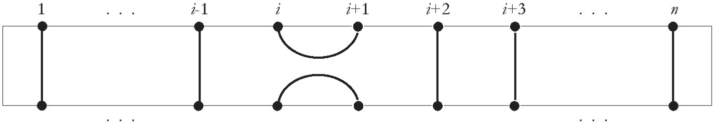

$= d_i$

Claim 1 and Claim 2, along with the fact that $\mathrm{TL}_n(\delta)$ and $\mathcal{D}_n(\delta)$ have the same dimension, suggest the following theorem.

Theorem

$\mathrm{TL}_n(\delta)$ and $\mathcal{D}_n(\delta)$ are isomorphic as $\mathbb{Z}[\delta]$-algebras under the correspondence

$$e_i \mapsto d_i.$$

Now, consider the group algebra of the symmetric group S_n over $\mathbb{Z}$:

$$\mathbb{Z}[S_n]$$

Recall that S_n is generated by the adjacent transpositions:

$$(1\ 2), (2\ 3), \ldots, (n-1\ \ n).$$

Define

$$s_i = (i\ \ i+1).$$

Next, take the principal ideal, J, of $\mathbb{Z}[S_n]$ generated by all elements of the form

$$1 + s_i + s_j + s_i s_j + s_j s_i + s_i s_j s_i,$$

where $|i-j| = 1$ (i.e., s_i and s_j are noncommuting generators).

Definition

Let $\sigma = s_{i_1} \dots s_{i_r} \in S_n$ be reduced. We say that σ is *fully commutative*, or *FC*, if any two reduced expressions for σ may be obtained from each other by repeated commutation of adjacent generators. In other words, σ has no reduced expression containing $s_i s_j s_i$ for $|i - j| = 1$.

Example

$s_1 s_2 s_4 s_1 = (1\ 2)(2\ 3)(4\ 5)(1\ 2)$ is a reduced expression for an element in S_5. This element is *not* FC.

$$s_1 s_2 s_4 s_1 = s_1 s_2 s_1 s_4$$

Now, let

$$b_{s_i} = (1 + s_i) + J \in \mathbb{Z}[S_n]/J.$$

Theorem

As a unital algebra, $\mathbb{Z}[S_n]/J$ *is generated by* $b_{s_1}, \ldots, b_{s_{n-1}}$.

Definition

If $\sigma = s_{i_1} \ldots s_{i_r}$ is reduced and FC, then

$$b_\sigma = b_{s_{i_1}} \ldots b_{s_{i_r}}$$

is a well-defined element of $\mathbb{Z}[S_n]/J$.

Theorem

The set $\{b_\sigma : \sigma \ FC\}$ *is a free* $\mathbb{Z}$*-basis for* $\mathbb{Z}[S_n]/J$.

That is, $\mathbb{Z}[S_n]/J$ has a basis indexed by the fully commutative elements of S_n.

If we let $\delta = 2$, we have the following result.

Theorem

The algebras $\mathbb{Z}[S_n]/J$ and $\mathrm{TL}_n(2)$ are isomorphic as $\mathbb{Z}$-algebras under the correspondence

$$b_{s_i} = (1 + s_i) + J \mapsto d_i.$$