On the cyclically fully commutative elements of Coxeter groups

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Definition

A Coxeter system (W, S) consists of a group W (called a Coxeter group) generated by a set S of involutions with presentation

$$W = \langle S : s^2 = 1, (st)^{m(s,t)} = 1 \rangle,$$

where $m(s,t) \geq 2$ for $s \neq t$.

Comment

Since s and t are involutions, the relation $(st)^{m(s,t)} = 1$ can be rewritten as

$$m(s,t) = 2 \implies st = ts \quad \} \text{ short braid relations}$$

$$\left.\begin{array}{lcl} m(s,t) = 3 & \implies & sts = tst \\ m(s,t) = 4 & \implies & stst = tsts \\ & \vdots & \end{array}\right\} \text{ long braid relations}$$

Definition

We can encode (W, S) with a unique Coxeter graph Γ having:

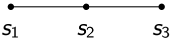
1. vertex set S;
2. edges $\{s, t\}$ labeled $m(s, t)$ whenever $m(s, t) \geq 3$ (typically labels with $m(s, t) = 3$ are omitted).

Comments

- W is irreducible if Γ is connected.
- Given Γ, we can reconstruct the corresponding (W, S).

Example

Coxeter graph of type A_3:



Then $W(A_3)$ is subject to: $s_1 s_2 s_1 = s_2 s_1 s_2$, $s_2 s_3 s_2 = s_3 s_2 s_3$, $s_1 s_3 = s_3 s_1$, and $s_i^2 = 1$.

Definition

A word $s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ is called an expression for $w \in W$ if it is equal to w when considered as a group element.

If m is minimal, it is a reduced expression, and the length of w is $\ell(w) := m$.

Example

Let $s_1 s_3 s_2 s_1 s_2$ be an expression for $w \in W(A_3)$. We see that

$$s_1 s_3 s_2 s_1 s_2 = s_1 s_3 s_1 s_2 s_1 = s_3 s_1 s_1 s_2 s_1 = s_3 s_2 s_1,$$

showing that the original expression is not reduced (and $\ell(w) = 3$).

Theorem (Matsumoto)

Any two reduced expressions for $w \in W$ differ by a sequence of braid relations.

Matsumoto's theorem provides an algorithmic solution to the word problem for Coxeter groups.

Conjugating an expression by an initial generator results in a cyclic shift of the word:

$$s_{x_1}(s_{x_1}s_{x_2}\cdots s_{x_m})s_{x_1} = s_{x_1}s_{x_1}s_{x_2}s_{x_3}\cdots s_{x_m}s_{x_1} = s_{x_2}s_{x_3}\cdots s_{x_m}s_{x_1}.$$

Definition

A reduced expression is conjugacy-reduced if every cyclic shift is reduced.

Question

Do two conjugacy-reduced expressions for conjugate group elements differ by a sequence of braid relations and cyclic shifts?

An affirmative answer would be a cyclic version of Matsumoto's theorem and would provide an algorithmic solution to the conjugacy problem for Coxeter groups.

Unfortunately, the answer is "no" ☹. Yet the answer is often "yes."

Goal

Find the largest subset for which the cyclic version of Matsumoto's theorem holds.

Definition

An element w is fully commutative (FC) if any two of its reduced expressions are equivalent by iterated short braid relations.

Theorem (Stembridge 1996)

w is FC iff every reduced expression "avoids long braid relations."

Example

A Coxeter element is an element for which every generator of S appears exactly once in each reduced expression. Clearly, Coxeter elements are FC.

Example

Let $s_1s_3s_2s_1$ be a reduced expressions for $w \in W(A_3)$. Then w is *not* FC since

$$s_1s_3s_2s_1 = s_3s_1s_2s_1.$$

Definition

An element w is cyclically fully commutative (CFC) if every cyclic shift of every reduced expression for w is a reduced expression for an FC element.

Comments

- The CFC elements are those whose "end-identified" reduced expressions avoid "collapse" and long braid relations.
- The CFC elements are the "cyclic version" of the FC elements.

Example

Clearly, Coxeter elements are CFC.

Example

Consider the reduced expression $s_2 s_1 s_3 s_2$ for $w \in W(A_3)$. Then w is FC, however, it is *not* CFC since it has a cyclic shift (involving s_2) that is not reduced:

$$s_1 s_3 s_2 s_2 = s_1 s_3.$$

If $s \in S$, then $\ell(sw) = \ell(w) \pm 1$, which implies that $\ell(w^k) \leq k \cdot \ell(w)$.

Definition (BBEEGM 2009)

An element $w \in W$ is logarithmic if $\ell(w^k) = k \cdot \ell(w)$ for all k.

Theorem (Speyer 2009)

In an infinite irreducible Coxeter group, Coxeter elements are logarithmic.

Theorem (H. Eriksson, K. Eriksson 2009)

The cyclic version of Matsumoto's theorem holds for Coxeter elements.

Theorem (BBEEGM 2009)

If W is an infinite irreducible Coxeter group with no odd $m(s,t)$ greater than 3, then the CFC elements having full support (i.e., every generator occurs in each reduced expression) are logarithmic.

Corollary (BBEEGM 2009)

Let W be an affine Weyl group (i.e., all $m(s,t) \in \{2,3,4,6,\infty\}$). If $w \in W$ is CFC with full support, then w is logarithmic and

1. *w is a Coxeter element, or*
2. *w^k is FC for all k.*

Conjecture ("Rabbit Hole of Death")

In an infinite irreducible Coxeter group, CFC elements with full support are logarithmic.

From here, we expect to be able to extend the Erikssons' techniques to establish the cyclic version of Matsumoto's theorem for these elements.

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