Diagram algebras and Kazhdan–Lusztig polynomials

Dana Ernst

University of Colorado at Boulder
Department of Mathematics
http://math.colorado.edu/~ernstd

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Definition

A *Coxeter group* is a group W together with a set S of generating involutions subject to defining relations

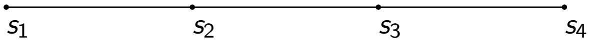
$$(s_i s_j)^{m_{ij}} = 1,$$

where $m_{ii} = 1$ (each generator is an involution) and $m_{ij} = m_{ji}$.

We can represent a Coxeter group using a *Coxeter graph* Γ:

- vertices of Γ are the elements of S
- connect s_i to s_j by an edge labeled m_{ij}, except we omit an edge if $m_{ij} = 2$, and if $m_{ij} = 3$, we omit the label.

Example



Coxeter graph of type A_4

The graph tells us that

1. If $|i - j| = 1$, then $(s_i s_j)^3 = 1$ iff $s_i s_j s_i = s_j s_i s_j$. These relations are referred to as *long braid relations*.
2. And if $|i - j| > 1$, then $(s_i s_j)^2 = 1$ iff s_i and s_j commute.

For example, $s_1 s_2 s_1 = s_2 s_1 s_2$ and $s_1 s_3 = s_3 s_1$.

In this case, the underlying Coxeter group W is isomorphic to the symmetric group S_5 under the correspondence

$$s_i \mapsto (i \ \ i+1) \in S_5.$$

Comment

In general, the underlying Coxeter group of type A_n (straight line Coxeter graph with n vertices and all edges having weight 3) is isomorphic to S_{n+1}.

Definition

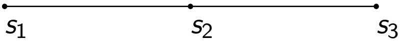
Every $w \in W$ can be written as a word in the generators:

$$w = s_{i_1} s_{i_2} \cdots s_{i_r}$$

If r is minimal, then we call this a *reduced expression* for w.
In this case, we define the *length* of w:

$$l(w) = r.$$

Example



Coxeter graph of type A_3

Let $w_1 = s_1 s_3 s_1 s_2 s_3 s_1$. This expression for w_1 is not reduced.

$$\begin{aligned} s_1 s_3 s_1 s_2 s_3 s_1 &= s_3 s_1 s_1 s_2 s_3 s_1 \\ &= s_3 s_1 s_1 s_2 s_3 s_1 \\ &= s_3 s_2 s_3 s_1 \end{aligned}$$

The last expression above is reduced. So, $l(w_1) = 4$. Notice that in the last reduced expression above, we have an opportunity to apply a long braid.

$$s_3 s_2 s_3 s_1 = s_2 s_3 s_2 s_1.$$

Example

Now, let $w_2 = s_2 s_1 s_3 s_2$. This is a reduced expression for w_2. So, $l(w_2) = 4$. However, we can apply one commutation.

$$s_2 s_1 s_3 s_2 = s_2 s_3 s_1 s_2.$$

These are the only reduced expressions for w_2. In particular, we never have an opportunity to apply a long braid relation.

Definition

We say that $w \in W$ is *fully commutative* if any two reduced expressions for w may be transformed into each other by iterated commutations.

Theorem

$w \in W$ is fully commutative iff no reduced expression for w contains a long braid.

Example

In the previous example, w_1 is not fully commutative since we were able to apply the long braid $s_3 s_2 s_3 = s_2 s_3 s_2$. However, w_2 is fully commutative.

Theorem

In a Coxeter group of type A_{n-1} ($W \cong S_n$), the number of fully commutative elements is equal to the nth Catalan number:

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}.$$

Example

In S_4, there are $4! = 24$ elements, of which

$$\frac{1}{5} \binom{8}{4} = 14$$

of these are fully commutative.

Hecke Algebras

Definition

Associated to a Coxeter group W, we have an associative $\mathbb{Z}[q, q^{-1}]$-algebra $\mathcal{H}_q$. This is a free module on the set $\{T_w : w \in W\}$, which satisfies

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } l(sw) > l(w), \\ q T_{sw} + (q-1) T_w, & \text{otherwise.} \end{cases}$$

This extends uniquely to an associative algebra structure. We extend the scalars to $\mathcal{A} := \mathbb{Z}[v, v^{-1}]$, where $v^2 = q$:

$$\mathcal{H} := \mathcal{A} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{H}_q.$$

We call $\mathcal{H}$ the *Hecke algebra* associated to W.

Comments

- If $w = s_{i_r} \cdots s_{i_r}$ (reduced), then

$$T_w = T_{s_{i_1}} \cdots T_{s_{i_r}}.$$

- $\mathcal{A}$ has a ring automorphism $\bar{\ }$ sending $v \mapsto v^{-1}$. This extends to a ring automorphism $\bar{\ } : \mathcal{H} \to \mathcal{H}$ satisfying

$$\overline{T_w} = (T_{w^{-1}})^{-1}.$$

  ($\bar{\ }$ is like inverse the revenge)
- Define $\widetilde{T_w} = v^{-l(w)} T_w$. Then $\{\widetilde{T_w} : w \in W\}$ is an $\mathcal{A}$-basis for $\mathcal{H}$.
- We define $\mathcal{L}$ to be the free $\mathbb{Z}[v^{-1}]$-module on the set $\widetilde{T_w}$. There exists a natural map $\pi : \mathcal{L} \to \mathcal{L}/v^{-1}\mathcal{L}$.

Theorem (Kazhdan, Lusztig)

There is a unique basis $\{C'_w : w \in W\}$ for $\mathcal{H}$ satisfying:

1. $\overline{C'_w} = C'_w$
2. $C'_w \in \mathcal{L}$ and $\pi(C'_w) = \pi\left(\widetilde{T_w}\right)$.

The basis $\{C'_w\}$ has important and subtle properties (like positivity properties).

Definition

The *Kazhdan–Lusztig polynomials* occur as follows. If

$$C'_w = \sum_{y \leq w} P^*_{y,w} \widetilde{T_y},$$

where $\leq$ is the Bruhat order on the Coxeter group W, then

$$P_{y,w} := v^{l(w)-l(y)} P^*_{y,w}.$$

Properties of K–L polynomials

1. $P_{w,w} = 1$ for all $w \in W$
2. $P_{y,w} \in \mathbb{Z}[q]$ (Acutally, $\mathbb{Z}_{\geq 0}[q]$ ... deep!)
3. $P_{y,w} = 0$ unless $y \leq w$
4. If $P_{y,w} \neq 0$, then $\deg P_{y,w} \leq \frac{1}{2}(l(w) - l(y) - 1)$
5. We write $\mu(y, w) \in \mathbb{Z}$ for the coefficient of $q^{1/2(l(w)-l(y)-1)}$ in $P_{y,w}$. Clearly, $\mu(y, w) = 0$ unless both $y < w$ and $l(w)$ and $l(y)$ have different parity.

Properties of K–L polynomials (continued)

6. There is a recursive formula

$$P_{x,w} = q^{1-c} P_{sx,v} + q^c P_{x,v} - \sum_{z \prec v, sz < z} \mu(z,w) q^{1/2(l(w)-l(z)-1)} P_{x,z},$$

where $sw = v < w$ and $c = \begin{cases} 0, & \text{if } x < sx \\ 1, & \text{otherwise.} \end{cases}$

Comment

Here's the upshot.

- There is natural basis indexed by the elements of W for $\mathcal{H}$: $\{T_w\}$.
- There is this another really nice basis that we like better: $\{C'_w\}$.
- The K–L polynomials essentially occur as the entries in the change of basis matrix from one basis to the other.
- The μ-values occur as the coefficients on the highest degree term in the corresponding K–L polynomial.
- Computing the K–L polynomials is a pain in the butt.
- Computing the μ-values is helpful, but not known to be any easier.

0–1 Conjecture

In S_n, $\mu(y, w)$ is always 0 or 1.

Theorem (Maclarnan, Warrington, 2003)

Conjecture fails in S_{10} and up.

Comment

Conjecture does hold for some special classes of elements.

Theorem

In S_n, if y is fully commutative, then $\mu(y, w)$ is always 0 or 1.

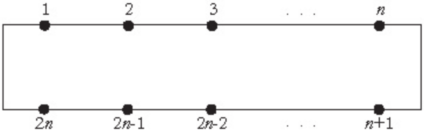
Current Research

There are quite a few people (like me) trying to find non-recursive ways to compute K–L polynomials and/or μ-values for various Coxeter groups.

Diagram algebras

Definition

A *standard n-box* is a rectangle with $2n$ nodes, labeled as follows:

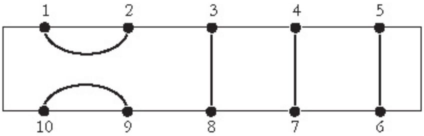


An *n-diagram* is a graph drawn on the nodes of a standard n-box such that

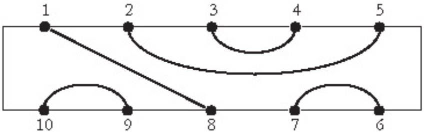
- Every node is connected to exactly one other node by a single edge.
- All edges must be drawn inside the n-box.
- The graph can be drawn so that no edges cross.

Example

Here is an example of a 5-diagram.

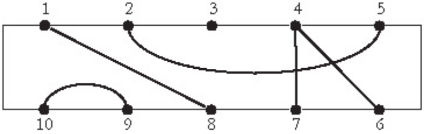


Here is another.



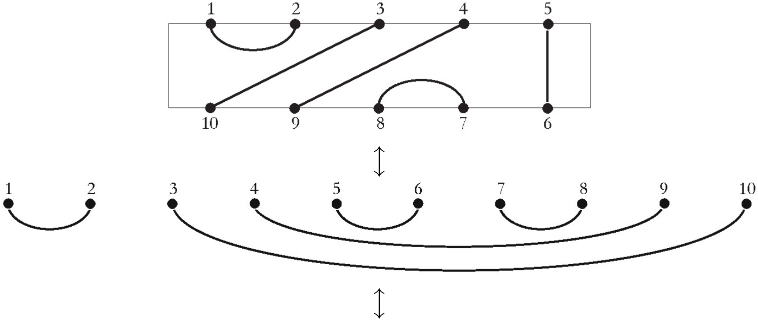
Example

Here is an example that is not a diagram.



Comment

There is a one-to-one correspondence between *n*-diagrams and sequences of *n* pairs of well-formed parentheses.



()((()()))

It is well-known that the number of sequences of *n* pairs of well-formed parentheses is C_n. Therefore, the number of *n*-diagrams is C_n.

Definition

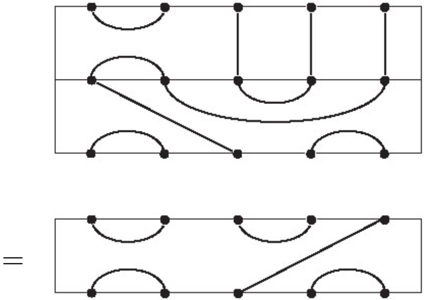
The *Temperley-Lieb algebra of type A*, $\mathrm{TL}_n(A)$, is the free $\mathcal{A}$-module having the set of n-diagrams as a basis with multiplication defined as follows.

If d and d' are n-diagrams, then dd' is obtained by identifying the "south face" of d with the "north face" of d', and then replacing any closed loops with a factor of $\delta = v + v^{-1}$.

TL_n is an associative algebra.

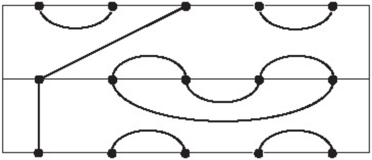
Example

Multiplication of two 5-diagrams.

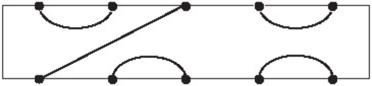


Example

Here's another example.

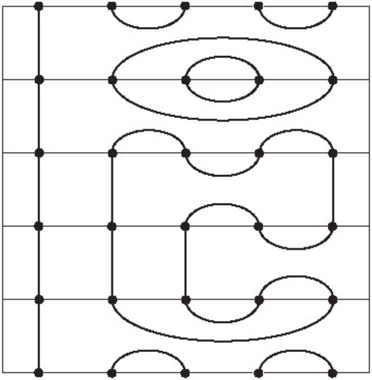


$= \delta$

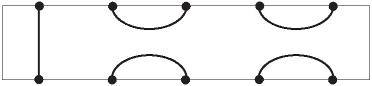


Example

And here's one more.

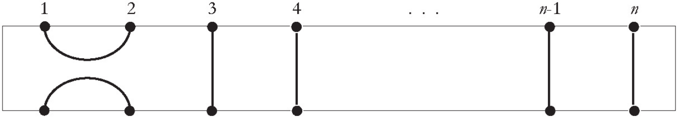


$= \delta^3$



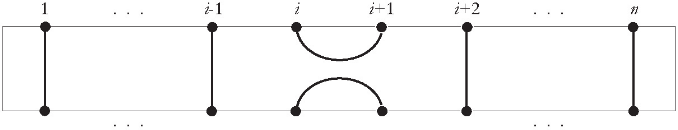
Now, we define a few "simple" n-diagrams. Let

$d_1 =$



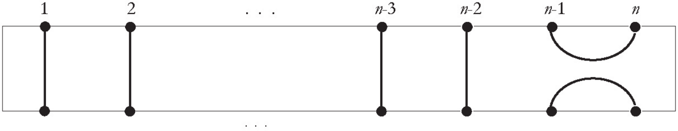
$\vdots$

$d_i =$



$\vdots$

$d_{n-1} =$



Claim

The set of "simple" diagrams generate TL_n as a unital algebra.

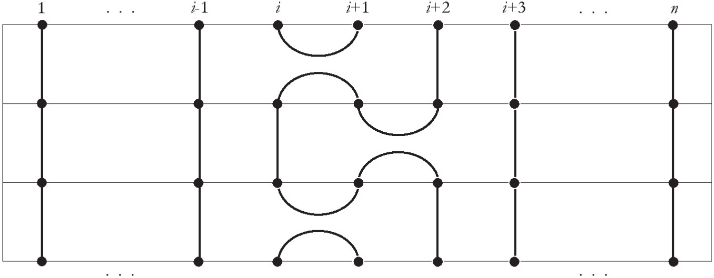
Theorem

TL_n has a presentation (as a unital algebra):

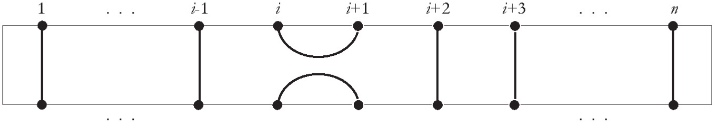
1. $d_i^2 = \delta d_i$, for all i
2. $d_i d_j = d_j d_i$, for $|i - j| \geq 2$
3. $d_i d_j d_i = d_i$, for $|i - j| = 1$

Here's the most interesting relation. The other two are also easy to check. For $|i-j|=1$ (here, $j=i+1$; $j=i-1$ being similar), we have

$d_i d_j d_i =$



$=$



$= d_i$

Comments

- $\mathrm{TL}_n(A)$ as an algebra with the presentation given above was invented in 1971 by Temperley and Lieb.
- First arose in the context of integrable Potts models in statistical mechanics.
- As well as having applications in physics, $\mathrm{TL}_n(A)$ appears in the framework of knot theory, braid groups, Coxeter groups and their corresponding Hecke algebras, and subfactors of von Neumann algebras.
- Penrose/Kauffman use diagram algebra to model $\mathrm{TL}_n(A)$ in 1971.
- In 1987, Vaughan Jones recognized that $\mathrm{TL}_n(A)$ is isomorphic to a particular quotient of the Hecke Algebra of type A_{n-1} (the symmetric group, S_n).

Theorem

TL_n is isomorphic to a quotient of the Hecke algebra of type A_{n-1} and has a basis indexed by the fully commutative elements of the underlying Coxeter group S_n. In particular, there exists a surjective homomorphism $\theta : \mathcal{H} \to \mathrm{TL}_n$, where

$$\theta\left(C'_{s_i}\right) = d_i.$$

Suppose $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ (reduced). Define $d_w = d_{i_1} d_{i_2} \cdots d_{i_r}$. Then

$$\theta\left(C'_w\right) = \begin{cases} d_w, & \text{if } w \text{ is fully commutative} \\ 0, & \text{otherwise.} \end{cases}$$

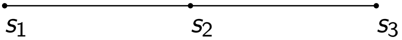
Theorem (R.M. Green)

If y and w are fully commutative elements of S_n, then $\mu(y, w)$ can be computed (non-recursively) as follows.

1. Draw diagrams for d_y and $d_{w^{-1}}$.
2. Multiply d_y times $d_{w^{-1}}$. Do not replace any closed loops with δ.
3. Connect point i in north face to point $2n - i + 1$ in south face (w/o intersections).

If this forms $n - 1$ closed loops, then $\mu(y, w) = 1$, and otherwise, $\mu(y, w) = 0$.

Example



Coxeter graph of type A_3

Let $y = s_2$ and $w = s_2 s_1 s_3 s_2$. Note that both y and w are fully commutative. We see that $w^{-1} = s_2 s_3 s_1 s_2$. Then

$$d_{w^{-1}} = d_2 d_3 d_1 d_2.$$

Finish on chalk board...

Closing Remarks

- What we are really doing when we "wrap up" $d_y d_{w^{-1}}$ is defining a trace function on a quotient of the Hecke algebra.
- Having a diagrammatic representation of this quotient allows us to easily define and compute this trace.
- This trace function is a generalized Jones trace and satisfies the Markov property.
- When this type of trace function is known to exist, we can use it to compute $\mu(y, w)$ for y and w fully commutative.
- At this point, only when we have a diagrammatic representation of the appropriate Hecke algebra quotient have we been able to define the trace that can be used to compute μ-values (types A, B, D, H, E, and $\widetilde{A}$).

Closing Remarks (continued)

- My Ph.D. thesis focuses on establishing a faithful representation of a generalized Temperley–Lieb algebra of type $\widetilde{C}$ by a particular diagram algebra.
- One application of this representation is a simple construction of a trace on the corresponding Hecke algebra, which can then be used to compute μ-values in a non-recursive way.
- This is the first successful attempt at this type of construction for a Coxeter group having an infinite number of fully commutative elements and a Coxeter graph involving edge weights greater than 3.