A diagrammatic representation of an affine $C$ Temperley–Lieb algebra

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Definition

The standard $k$-box is a rectangle w/ $2k$ nodes labeled as:

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  2 4 6 8
  1 3 5 7
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A concrete $k$-diagram consists of a finite number of disjoint curves (planar), called edges, embedded in and disjoint from the box s.t.:

1. edges may be isotopic to circles, but not if their endpoints coincide w/ the nodes of the box,
2. the nodes of the box are the endpoints of curves, which meet the box transversely.

An edge joining $i$ in the N-face to $j$ in the S-face is called a propagating edge.

Two concrete diagrams are equivalent if one concrete diagram can be obtained from the other by isotopically deforming the edges s.t. any intermediate diagram is also a concrete diagram.

A $k$-diagram is defined to be an equivalence class of equivalent concrete $k$-diagrams.

Examples

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A concrete 5-diagram

Not a concrete 5-diagram
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Definition (continued)

Let $d$ be a concrete $(n+2)$-diagram w/ edge $e$. We may adorn $e$ w/ a finite sequence of blocks of decorations from $\mathcal{V}$ if adjacency of blocks and decorations is preserved as we travel along $e$. Each decoration on $e$ has an associated height, called its vertical position. $d$ is a concrete LR-decorated diagram if it satisfies:

1. if $d$ has no non-prop edges, then $d$ is undecorated;
2. it is possible to deform all decorated edges so as to take open decorations to the left and closed decorations to the right simultaneously;
3. if $d$ is non-prop, then we allow adjacent blocks on $e$ to be conjoined to form larger blocks;
4. if $d$ has more than 1 non-prop edge in N-face and $e$ is prop, then we allow adjacent blocks on $e$ to be conjoined to form larger blocks;
5. if $d$ has exactly one non-prop edge in N-face and $e$ is prop, then:
   a. all decorations occurring on prop edges must have vertical position lower (resp. higher) than the vertical positions of decorations occurring on the (unique) non-prop edge in the N-face (resp. S-face);
   b. if $d$ is block occurring on $e$, then no other decorations occurring on any other prop edge may have vertical position in the range of vertical positions that $b$ occupies;
   c. if $b$ and $b'$ are two adjacent blocks occurring on $e$, they may be conjoined to form a larger block only if the previous requirement is not violated.

Two concrete LR-decorated diagrams are $\mathcal{V}$-equivalent if we can isotopically deform one diagram into the other s.t. any intermediate diagram is also a concrete LR-decorated diagram.

An LR-decorated diagram is an equivalence class of $\mathcal{V}$-equivalent concrete LR-decorated diagrams.

Examples

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Two examples of LR-decorated 5-diagrams

Example of LR-decorated 6-diagram
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Definition

Let $P_{n}^{LR}(\mathcal{V})$ be the free $\mathcal{Z}[\mathcal{V}]$-module w/ basis consisting of the LR-decorated diagrams having blocks that do not contain any adjacent decorations of the same type (open/closed) and do not have any of the loops listed below.

To calculate $d'$, concatenate $d$ and $d'$. While maintaining $\mathcal{V}$-equivalence, conjoin adjacent blocks subject to:

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Examples

Examples of multiplication in $P_{4}^{LR}(\mathcal{V})$

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Definition (continued)

The simple diagrams $d_{1}, d_{2}, \ldots, d_{n+1}$ of $P_{n}^{LR}(\mathcal{V})$ are:

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Let $D_{n}$ be the $\mathcal{Z}[\mathcal{V}]$-subalgebra of $P_{n}^{LR}(\mathcal{V})$ generated by the simple diagrams.

Definition

An LR-decorated diagram $d$ is admissible if it satisfies:

1. The only loops that appear are equivalent to $\delta$.
2. If $d$ has no prop edges, then the edges joining 1 and 1' (resp, n and n') are decorated w/ a non-prop edge only by a single block.
3. If $d$ has exactly one prop edge $e$, $e$ is decorated by an alternating sequence (possibly empty) of $\mathcal{A}$ and $\mathcal{D}$. If $e$ is connected to 1 (resp, n), then the highest (resp, lowest) decoration occurring on $e$ is $\mathcal{A}$. If $e$ is connected to 1' (resp, n'), then the highest (resp, lowest) decoration occurring on $e$ is $\mathcal{D}$. If there is no non-prop edge connected to 1 or 1' (resp, n or n'), then $d$ is decorated by a single $\mathcal{A}$ (resp, $\mathcal{D}$).
4. If $d$ has exactly one prop edge in the N-face, then the leftmost prop edge is equal to one of the following, where the rectangle represents a sequence of blocks (possibly empty), where each block is a single $\mathcal{A}$.

Examples

Examples of multiplication in $P_{4}^{LR}(\mathcal{V})$

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Theorem (Ernst [1])

The admissible diagrams form a basis for $D_{n}$.

Definition

The Temperley–Lieb algebra of type affine $C$, denoted $TL(\mathcal{C})$, is the $\mathcal{Z}[\mathcal{V}]$-algebra generated as a unital algebra by $b_{1}, b_{2}, \ldots, b_{n+1}$ w/ defining relations:

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Theorem (Ernst [1])

The map $\theta : TL(\mathcal{C}) \to D_{n}$ given by $\theta(b_{i}) = d_{i}$ is an algebra isomorphism. Moreover, the admissible diagrams are in bijection w/ the monomial basis elements (see [3]) of $TL(\mathcal{C})$.

Applications and Current Research

We perform a change of basis to obtain a basis that coincides w/ the canonical basis of [4]. Using new representation, we define a trace on $\theta(\mathcal{C})$ and use it to non-recursively compute leading coefficients of certain Kazhdan–Lusztig polynomials (notoriously difficult to compute).

References