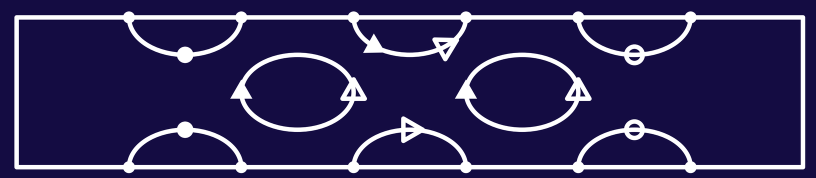
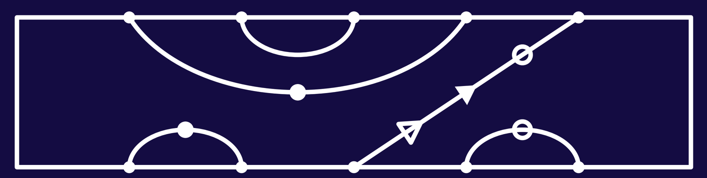


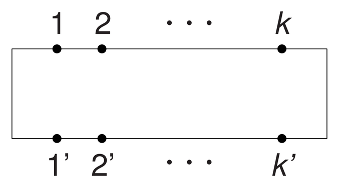
A diagrammatic representation of an affine C Temperley–Lieb algebra



Dana C. Ernst, Plymouth State University

Definition

The standard k-box is a rectangle w/ $2k$ nodes labeled as:



A concrete k-diagram consists of a finite number of disjoint curves (planar), called edges, embedded in and disjoint from the box s.t.

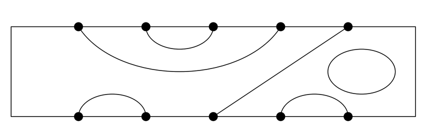
1. edges may be isotopic to circles, but not if their endpoints coincide w/ the nodes of the box;
2. the nodes of the box are the endpoints of curves, which meet the box transversely.

An edge joining i in the N-face to j' in the S-face is called a propagating edge.

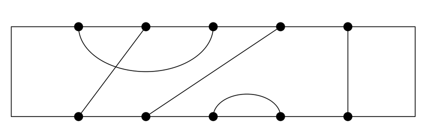
Two concrete diagrams are equivalent if one concrete diagram can be obtained from the other by isotopically deforming the edges s.t. any intermediate diagram is also a concrete diagram.

A k-diagram is defined to be an equivalence class of equivalent concrete k-diagrams.

Examples



A concrete 5-diagram



Not a concrete 5-diagram

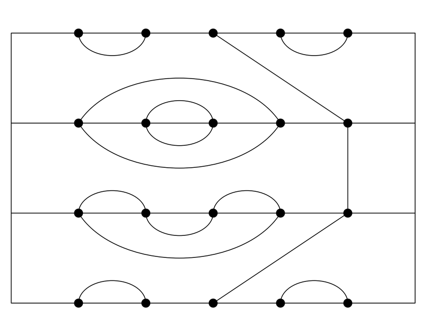
Definition

The (ordinary) Temperley–Lieb diagram algebra, denoted $\mathbb{D}\mathrm{TL}(A_n)$, is the free $\mathbb{Z}[\delta]$-module w/ basis consisting of the loop-free $(n+1)$-diagrams.

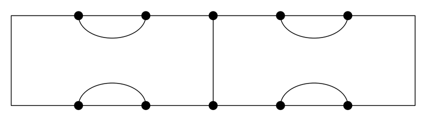
If d and d' are basis elmts, calculate the product dd' by identifying the S-face of d w/ the N-face of d' and then multiplying by a factor of δ for each loop and discard loop.

$\mathbb{D}\mathrm{TL}(A_n)$ is an assoc $\mathbb{Z}[\delta]$-algebra having the loop-free $(n+1)$-diagrams as a basis.

Example



$= \delta^3$



Example of multiplication in $\mathbb{D}\mathrm{TL}(A_4)$

Definition

We now describe a diagram algebra where the diagrams are allowed to carry decorations. Our decoration set is $\mathcal{V} = \{\bullet, \blacktriangle, \circ, \vartriangle\}$, where the first two decorations are called closed, and the last two are open. Any finite sequence of decorations is called a block. Fix $n \geq 2$.

Definition (continued)

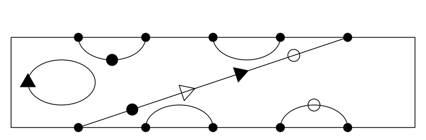
Let d be a concrete $(n+2)$-diagram w/ edge e. We may adorn e w/ a finite sequence of blocks of decorations from $\mathcal{V}$ if adjacency of blocks and decorations is preserved as we travel along e. Each decoration on e has an associated height, called its vertical position. d is a concrete LR-decorated diagram if it satisfies:

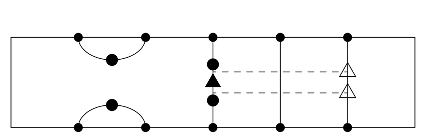
1. if d has no non-prop edges, then d is undecorated;
2. it is possible to deform all decorated edges so as to take open decorations to the left and closed decorations to the right simultaneously;
3. if e is non-prop, then we allow adjacent blocks on e to be conjoined to form larger blocks;
4. if d has more than 1 non-prop edge in N-face and e is prop, then we allow adjacent blocks on e to be conjoined to form larger blocks;
5. if d has exactly one non-prop edge in N-face and e is prop, then:
   a. all decorations occurring on prop edges must have vertical position lower (resp, higher) than the vertical positions of decorations occurring on the (unique) non-prop edge in the N-face (resp, S-face);
   b. if **b** is block occurring on e, then no other decorations occurring on any other prop edge may have vertical position in the range of vertical positions that **b** occupies;
   c. if $\mathbf{b}_i$ and $\mathbf{b}_{i+1}$ are two adjacent blocks occurring on e, they may be conjoined to form a larger block only if the previous requirement is not violated.

Two concrete LR-decorated diagrams are $\mathcal{V}$-equivalent if we can isotopically deform one diagram into the other s.t. any intermediate diagram is also a concrete LR-decorated diagram.

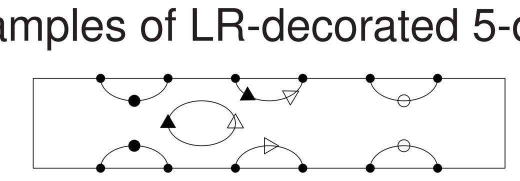
An LR-decorated diagram is an equivalence class of $\mathcal{V}$-equivalent concrete LR-decorated diagrams.

Examples





Two examples of LR-decorated 5-diagrams

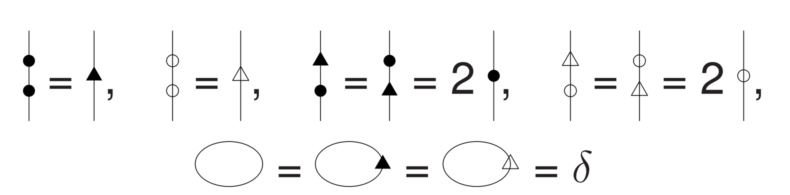


Example of LR-decorated 6-diagram

Definition

Let $\widehat{\mathcal{P}}^{LR}_{n+2}(\mathcal{V})$ be the free $\mathbb{Z}[\delta]$-module w/ basis consisting of the LR-decorated diagrams having blocks that do not contain any adjacent decorations of the same type (open/closed) and do not have any of the loops listed below.

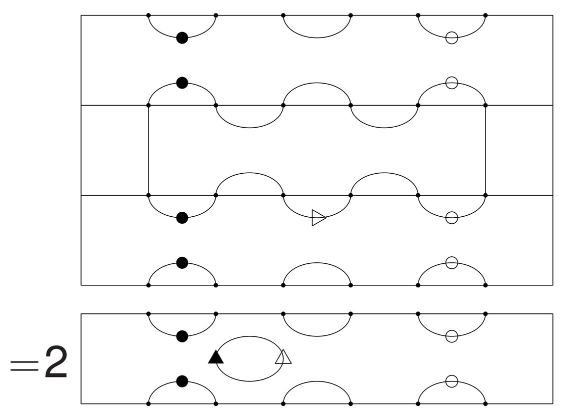
To calculate dd', concatenate d and d'. While maintaining $\mathcal{V}$-equivalence, conjoin adjacent blocks subject to:

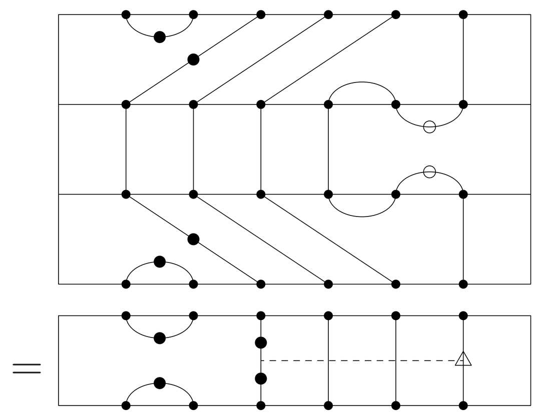


Theorem (Ernst [1])

$\widehat{\mathcal{P}}^{LR}_{n+2}(\mathcal{V})$ is an assoc $\mathbb{Z}[\delta]$-algebra. A basis consists of the LR-decorated diagrams having blocks that do not contain any adjacent decorations of the same type (open/closed) and there are no loops that can be replaced w/ δ.

Examples

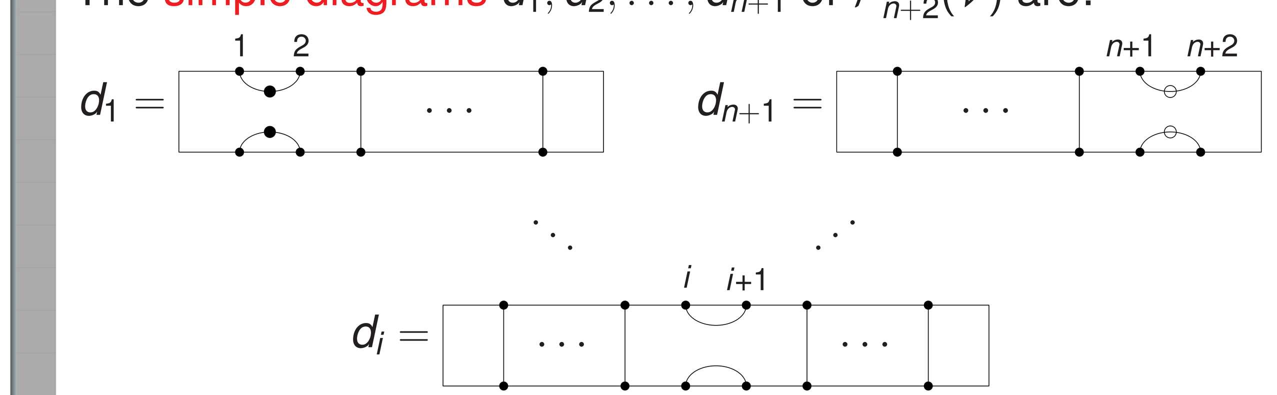




Examples of multiplication in $\widehat{\mathcal{P}}^{LR}_{n+2}(\mathcal{V})$

Definition

The simple diagrams $d_1, d_2, \ldots, d_{n+1}$ of $\widehat{\mathcal{P}}^{LR}_{n+2}(\mathcal{V})$ are:

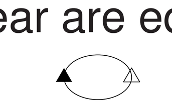


Let $\mathbb{D}_n$ be the $\mathbb{Z}[\delta]$-subalgebra of $\widehat{\mathcal{P}}^{LR}_{n+2}(\mathcal{V})$ generated by the simple diagrams.

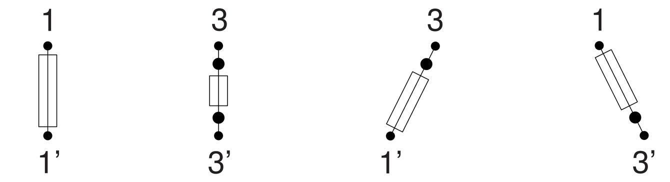
Definition

An LR-decorated diagram d is admissible if it satisfies:

1. The only loops that appear are equivalent to



2. If d has no prop edges, then the edges joining 1 and $1'$ (resp, $n+2$ and $(n+2)'$) are decorated w/ $\bullet$ (resp, $\circ$). These are the only $\bullet$ (resp, $\circ$) decorations occurring on d and are the leftmost (resp, rightmost) decorations on their resp edges.
3. If d has exactly one prop edge e, e is decorated by an alternating sequence (possibly empty) of $\blacktriangle$ and $\vartriangle$. If e is connected to 1 (resp, $1'$), then the highest (resp, lowest) decoration occurring on e is $\bullet$. If e is connected to $n+2$ (resp, $(n+2)'$), then the highest (resp, lowest) decoration occurring on e is $\circ$. If there is a non-prop edge connected to 1 or $1'$ (resp, $n+2$ or $(n+2)'$) it is decorated only by a single $\bullet$ (resp, $\circ$). No other $\bullet$ or $\circ$ appear.
4. If d has exactly one non-prop edge in the N-face, then the leftmost prop edge is equal to one of the following, where the rectangle represents a sequence of blocks (possibly empty), where each block is a single $\blacktriangle$.



Definition (continued)

The occurrences of the $\bullet$ decorations occurring on the prop edge are the highest or lowest decorations occurring on any prop edge. We have an analogous requirement for the rightmost prop edge w/ open. If there is a non-prop edge connected to 1 or $1'$ (resp, $n+2$ or $(n+2)'$) it is decorated only by a single $\bullet$ (resp, $\circ$). No other $\bullet$ or $\circ$ appear.

5. Assume that d has more than one non-prop edge and more than one prop edge. If e joins 1 to $1'$ (resp, $n+2$ to $(n+2)'$), then it is decorated by a single $\blacktriangle$ (resp, $\vartriangle$). Otherwise, an edge joining only one of 1 or $1'$ (resp, $n+2$ or $(n+2)'$) is decorated by a single $\bullet$ (resp, $\circ$) and no other $\bullet$ or $\circ$ appear.

Theorem (Ernst [1])

The admissible diagrams form a basis for $\mathbb{D}_n$.

Definition

The Temperley–Lieb algebra of type affine C, denoted $\mathrm{TL}(\widetilde{C}_n)$, is the $\mathbb{Z}[\delta]$-algebra generated as a unital algebra by $b_1, b_2, \ldots, b_{n+1}$ w/ defining relations

1. $b_i^2 = \delta b_i$ for all i,
2. $b_i b_j = b_j b_i$ if $|i-j| > 1$,
3. $b_i b_j b_i = b_i$ if $|i-j| = 1$ and $1 < i, j < n+1$,
4. $b_i b_j b_i b_j = 2 b_i b_j$ if $\{i,j\} = \{1,2\}$ or $\{n, n+1\}$.

Comments

1. $\mathrm{TL}(\widetilde{C}_n)$ is an infinite dim assoc algebra having a basis indexed by the fully commutative elmts of the Coxeter group of type $\widetilde{C}_n$. By [5], w in a Coxeter group W is fully commutative iff no reduced expression for w contains a long braid as a consecutive subexpression.
2. $\mathrm{TL}(\widetilde{C}_n)$ is a quotient of the Hecke algebra $\mathcal{H}(\widetilde{C}_n)$ [2].

Theorem (Ernst [1])

The map $\theta : \mathrm{TL}(\widetilde{C}_n) \to \mathbb{D}_n$ given by $\theta(b_i) = d_i$ is an algebra isomorphism. Moreover, the admissible diagrams are in bijection w/ the monomial basis elmts (see [3]) of $\mathrm{TL}(\widetilde{C}_n)$.

Applications and Current Research

We perform a change of basis to obtain a basis that coincides w/ the canonical basis of [4]. Using new representation, we define a trace on $\mathcal{H}(\widetilde{C}_n)$ and use it to non-recursively compute leading coefficients of certain Kazhdan–Lusztig polynomials (notoriously difficult to compute).

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[3] R.M. Green. Star reducible Coxeter groups. *Glasgow Math. J.*, 48:583–609, 2006.

[4] R.M. Green and J. Losonczy. Canonical bases for Hecke algebra quotients. *Math. Res. Lett.*, 6:213–222, 1999.

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