Diagram calculus for the Temperley–Lieb algebra

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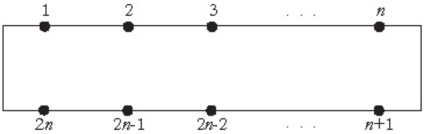
Diagram algebras

Definition

A *standard n-box* is a rectangle with $2n$ nodes, labeled as follows:

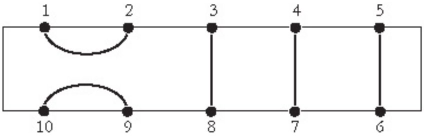


An *n-diagram* is a graph drawn on the nodes of a standard n-box such that

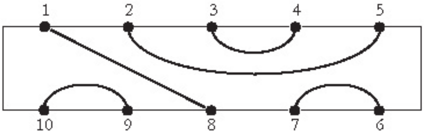
- Every node is connected to exactly one other node by a single edge.
- All edges must be drawn inside the n-box.
- The graph can be drawn so that no edges cross.

Example

Here is an example of a 5-diagram.

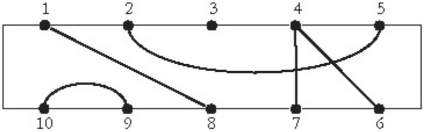


Here is another.



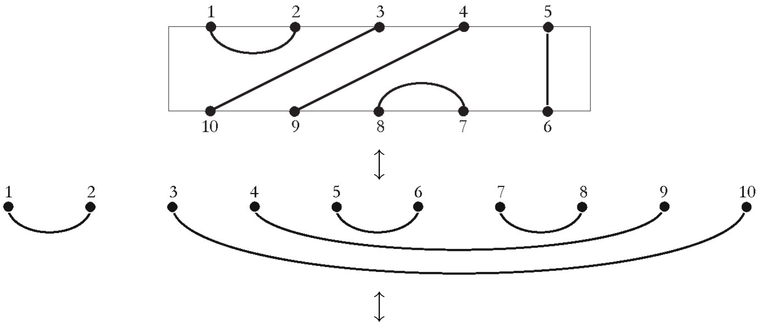
Example

Here is an example that is not a diagram.



Comment

There is a one-to-one correspondence between n-diagrams and sequences of n pairs of well-formed parentheses.



$$()((()()))$$

It is well-known that the number of sequences of n pairs of well-formed parentheses is equal to the nth Catalan number. Therefore, the number of n-diagrams is equal to the nth Catalan number.

Comment (continued)

- The *n*th Catalan number is given by

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}.$$

- The first few Catalan numbers are $1, 1, 2, 5, 14, 42, 132$.
- Richard Stanley's book, "Enumerative Combinatorics, Vol II," contains 66 different combinatorial interpretations of the Catalan numbers. An addendum online includes additional interpretations for a grand total of 161 examples of things that are counted by the Catalan numbers.
- In this talk, we'll see one more example of where the Catalan numbers turn up.

Definition

The Temperley-Lieb algebra, $\mathrm{TL}_n(\delta)$, with parameter δ is the free $\mathbb{Z}[\delta]$-module having the set of n-diagrams as a basis with multiplication defined as follows.

If d and d' are n-diagrams, then dd' is obtained by identifying the "south face" of d with the "north face" of d', and then replacing any closed loops with a factor of δ.

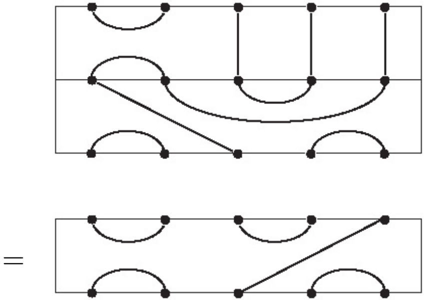
TL_n is an associative algebra. That is, the multiplication of n-diagrams is associative.

Comment

- $\mathbb{Z}[\delta]$ is the set of all polynomials in δ with integer coefficients. For example,
$$\delta^3 - 4\delta + 1 \in \mathbb{Z}[\delta].$$
- In this context, we should think of an algebra as being like a vector space, except we can also multiply the "vectors," which in this case are diagrams. Also, everything here is happening over $\mathbb{Z}[\delta]$ instead of a field.
- A typical element of $\mathrm{TL}_n(\delta)$ looks like a linear combination of n-diagrams, where the coefficients in the linear combination are polynomials in δ.
- Let's look at some examples of multiplication of diagrams.

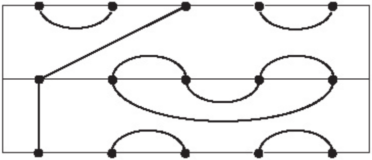
Example

Multiplication of two 5-diagrams.

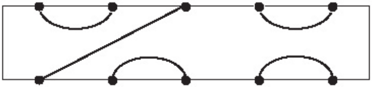


Example

Here's another example.

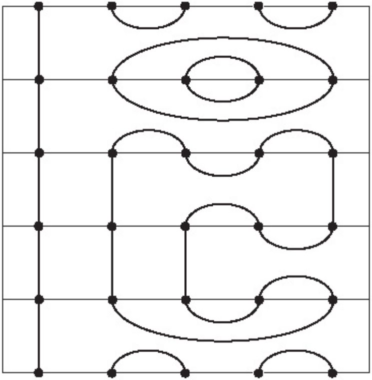


$= \delta$

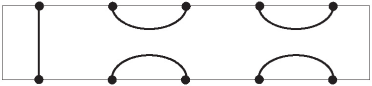


Example

And here's one more.



$= \delta^3$



Theorem

In general, the product of any number of n-diagrams will be equal to

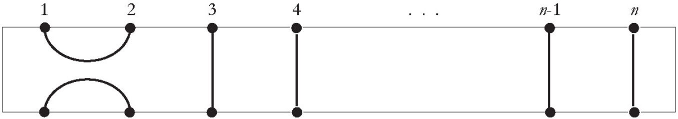
δ^k | *some n-diagram*

where $0 \leq k < \infty$*. Note that* $k = 0$ *if there are no loops in the product.*

Now, we define a few "simple" *n*-diagrams. These diagrams will form a generating set for $\mathrm{TL}_n(\delta)$.

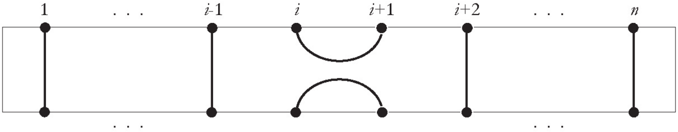
Let

$d_1 =$



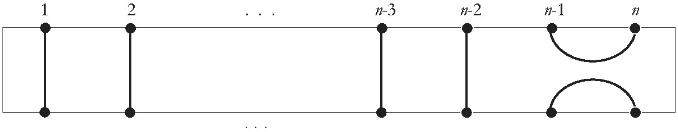
$\vdots$

$d_i =$



$\vdots$

$d_{n-1} =$



Claim

The set of "simple" diagrams generate $\mathrm{TL}_n(\delta)$ as a unital algebra.

In this case, we can write any n-diagram as a product of the "simple" n-diagrams.

Theorem

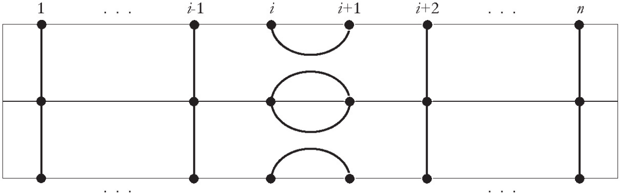
$\mathrm{TL}_n(\delta)$ has a presentation (as a unital algebra):

1. $d_i^2 = \delta d_i$, for all i
2. $d_i d_j = d_j d_i$, for $|i - j| \geq 2$
3. $d_i d_j d_i = d_i$, for $|i - j| = 1$

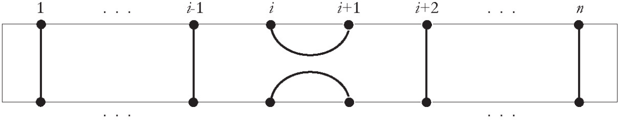
Let's check that these relations actually hold.

For all i, we have

$d_i^2 =$



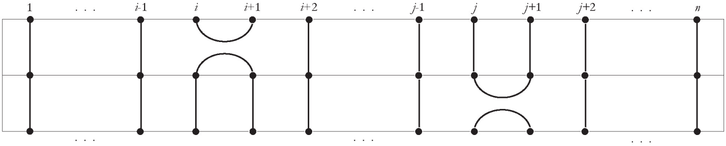
$= \delta$



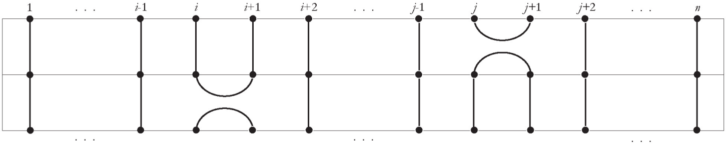
$= \delta d_i$

For $|i - j| \geq 2$, we have

$d_i d_j =$



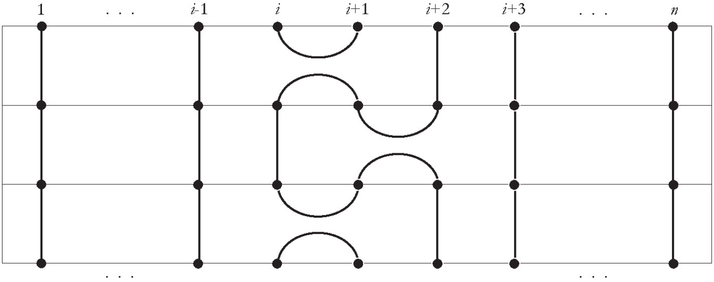
$=$



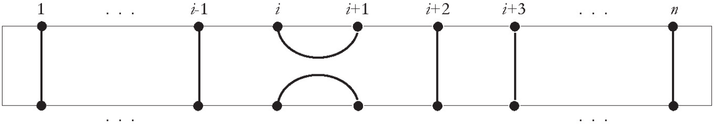
$= d_j d_i$

For $|i - j| = 1$ (here, $j = i + 1$; $j = i - 1$ being similar), we have

$d_i d_j d_i =$



$=$



$= d_i$

Comments

- $\mathrm{TL}_n(\delta)$ as an algebra with the presentation given above was invented in 1971 by Temperley and Lieb.
- First arose in the context of integrable Potts models in statistical mechanics.
- As well as having applications in physics, $\mathrm{TL}_n(\delta)$ appears in the framework of knot theory, braid groups, Coxeter groups and their corresponding Hecke algebras, and subfactors of von Neumann algebras.
- Penrose/Kauffman used diagram algebra to model $\mathrm{TL}_n(\delta)$ in 1971.
- In 1987, Vaughan Jones (awarded Fields Medal in 1990) recognized that $\mathrm{TL}_n(\delta)$ is isomorphic to a particular quotient of the Hecke algebra of type A_{n-1} (the Coxeter group of type A_{n-1} is the symmetric group, S_n).

Example

$\mathrm{TL}_3(\delta)$ is generated by d_1 and d_2, where these generators satisfy the relations

$$d_1^2 = \delta d_1 \text{ and } d_2^2 = \delta d_2$$
$$d_1 d_2 d_1 = d_1 \text{ and } d_2 d_1 d_2 = d_2$$

Example

$\mathrm{TL}_4(\delta)$ is generated by d_1, d_2, and d_3 where these generators satisfy the relations

$$d_1^2 = \delta d_1, d_2^2 = \delta d_2, \text{ and } d_3^2 = \delta d_3$$
$$d_1 d_3 = d_3 d_1$$
$$d_1 d_2 d_1 = d_1 \text{ and } d_2 d_1 d_2 = d_2$$
$$d_2 d_3 d_2 = d_2 \text{ and } d_3 d_2 d_3 = d_2$$

Theorem

A basis for TL_n may be described in terms of "reduced words" in the algebra generators d_i.

Example

Consider the following expression in $\mathrm{TL}_4(\delta)$.

$$d_1 d_3 d_1 d_2 d_3.$$

This expression is not "reduced".

Example (continued)

$$\begin{aligned} d_1 d_3 d_1 d_2 d_3 &= d_3 d_1 d_1 d_2 d_3 \\ &= d_3 d_1 d_1 d_2 d_3 \\ &= \delta d_3 d_1 d_2 d_3 \\ &= \delta d_3 d_1 d_2 d_3 \\ &= \delta d_1 d_3 d_2 d_3 \\ &= \delta d_1 d_3 d_2 d_3 \\ &= \delta d_1 d_3 \end{aligned}$$

The expression $d_1 d_3$ is "reduced" and represents a basis element of $\mathrm{TL}_3(\delta)$. Note that it's not the only reduced expression for this basis element.

$$d_1 d_3 = d_3 d_1$$

The symmetric group S_n

Now, let's consider the symmetric group, S_n. Recall that S_n is generated by the adjacent transpositions:

$$(1\ 2), (2\ 3), \ldots, (n-1\ \ n).$$

That is, every element of S_n can be written as a product of the adjacent transpositions.

Now, define

$$s_i = (i\ \ i+1).$$

Example

S_4 is generated by

$$s_1 = (1\ 2), s_2 = (2\ 3), s_3 = (3\ 4).$$

Comment

Note that S_n satisfies the following relations:

1. $s_i^2 = 1$ for all i (transpositions are order 2)
2. $s_i s_j = s_j s_i$, for $|i - j| \geq 2$ (disjoint cycles commute)
3. $s_i s_j s_i = s_j s_i s_j$, for $|i - j| = 1$ (called the braid relations)

In fact, we can use these relations to define S_n. Also, notice that these relations look similar to the relations of $\mathrm{TL}_n(\delta)$.

Comment (continued)

Every element of S_n can be written as a word in these generators and we can use the relations to potentially decrease the number of generators occurring in a word.

Example

In S_4

$$(1\ 2\ 3\ 4) = (1\ 2)(2\ 3)(3\ 4) = s_1 s_2 s_3.$$

This is an example of a "reduced" word in S_4. However, the expression

$$s_1 s_3 s_1 s_2 s_3 s_1$$

is not a reduced word.

$$\begin{aligned} s_1 s_3 s_1 s_2 s_3 s_1 &= s_3 s_1 s_1 s_2 s_3 s_1 \\ &= s_3 s_1 s_1 s_2 s_3 s_1 \\ &= s_3 s_2 s_3 s_1 \end{aligned}$$

Example (continued)

The last expression above is reduced. Notice that we could apply a braid relation in the last expression above, but it does not reduce the last expression above.

$$s_3 s_2 s_3 s_1 = s_2 s_3 s_2 s_1$$

We can also commute s_1 and s_3, but this does not reduce the word either.

$$s_3 s_2 s_3 s_1 = s_3 s_2 s_1 s_3$$

Definition

Let $\sigma = s_{i_1} \dots s_{i_r} \in S_n$ be reduced. We say that σ is fully commutative, or FC, if any two reduced expressions for σ may be obtained from each other by repeated commutation of adjacent generators. In other words, σ has no reduced expression containing $s_i s_j s_i$ for $|i - j| = 1$ (that is, there are no opportunities to apply a braid relation).

Example

In the previous example, $s_1 s_2 s_3$ is FC. However, $s_3 s_2 s_3 s_1$ is not FC because we have an opportunity to apply a braid relation.

A group algebra of S_n

Now, consider the group algebra of the symmetric group S_n over $\mathbb{Z}$:

$$\mathbb{Z}[S_n]$$

This algebra consists of linear combinations of reduced words in the generators $s_1, \ldots, s_{n-1}$, where the coefficients in the linear combination are integers. For example,

$$s_1 s_2 + 3 s_2 s_3 s_2 \in \mathbb{Z}[S_4].$$

Comment

The elements of S_n form a free $\mathbb{Z}$-basis for $\mathbb{Z}[S_n]$.

Next, take the two-sided ideal, J, of $\mathbb{Z}[S_n]$ generated by all elements of the form

$$1 + s_i + s_j + s_i s_j + s_j s_i + s_i s_j s_i,$$

where $|i - j| = 1$ (i.e., s_i and s_j are noncommuting generators).

Example

Consider $\mathbb{Z}[S_3]$. In this case, J is generated by

$$1 + s_1 + s_2 + s_1 s_2 + s_2 s_1 + s_1 s_2 s_1.$$

What this means is that J is the smallest ideal containing the linear combination above (it is closed under multiplication on the left and right by $\mathbb{Z}$-linear combinations of elements from S_n).

Now, we consider the quotient algebra $\mathbb{Z}[S_n]/J$. Let

$$b_{s_i} = (1 + s_i) + J \in \mathbb{Z}[S_n]/J.$$

Definition

If $\sigma = s_{i_1} \dots s_{i_r}$ is reduced and FC, then

$$b_\sigma = b_{s_{i_1}} \dots b_{s_{i_r}}$$

is a well-defined element of $\mathbb{Z}[S_n]/J$. b_σ for σ FC is called a monomial.

Theorem

As a unital algebra, $\mathbb{Z}[S_n]/J$ is generated by $b_{s_1}, \ldots, b_{s_{n-1}}$. Furthermore, the set $\{b_\sigma : \sigma\ FC\}$ is a free $\mathbb{Z}$-basis for $\mathbb{Z}[S_n]/J$.

That is, $\mathbb{Z}[S_n]/J$ has a basis indexed by the fully commutative elements of S_n. We should think of $\mathbb{Z}[S_n]/J$ as the set of all linear combinations of monomials (indexed by FC elements of S_n), where the coefficients of the linear combination are integers.

If we let $\delta = 2$, we have the following result.

Theorem

The algebras $\mathbb{Z}[S_n]/J$ and $\mathrm{TL}_n(2)$ are isomorphic as $\mathbb{Z}$-algebras under the correspondence

$$b_{s_i} = (1 + s_i) + J \mapsto d_i.$$

That is, the quotient algebra $\mathbb{Z}[S_n]/J$ can be represented by the diagram algebra that we introduced earlier, where we set $\delta = 2$.

Corollary

Therefore, the number of FC elements in S_n is equal to the nth Catalan number.