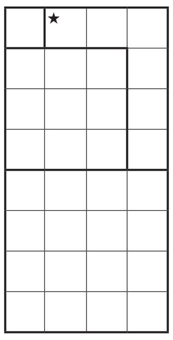
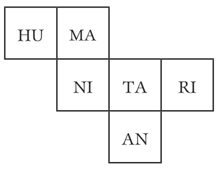
OCTOBER

NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Magic squares: Record solutions in this answer box. Begin in the starred box; proceed left to right in each row. Answers form two magic squares, a 3×3 and a 4×4 array. Remaining answers pay tribute to a master of recreational mathematics.

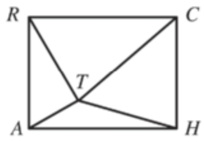


1 The six squares in the diagram are to be folded into a cube. A humanitarian wants to know which side will be opposite the side labeled AN. Can you help him determine the answer?



2 *CHAR* is a rectangle with point *T* in its interior. Determine which line segment in the diagram has the square of its measure equal to the expression

$$(AT)^2 - (HT)^2 + (CT)^2.$$



3 Each letter of the alphabet is assigned an integer, starting with A = 0, B = 1, and so on. The numbers repeat after every seven letters, so that G = 6, H = 0, and I = 1, continuing on to Z. What two-letter word is represented by the digits 16?

4 The graphs of the equations

$$y = 3^x \text{ and } y = 12x - 9$$

intersect. Find the sum of the x-values for all points of intersection.

5 Only 2 two-digit integers are equal to three times the product of their digits. Find the absolute value of the difference of these 2 integers.

6 Evaluate

$$\frac{\binom{2011}{1006}}{\binom{2010}{1005}} = \frac{{}_{2011}C_{1006}}{{}_{2010}C_{1005}}$$

to the nearest integer.

7 *GUIDANCE* is a regular octagon with center *O*. We define R_1, R_2, and R_3 as counterclockwise rotations of, respectively, 45°, 90°, and 135° about *O*. Which diagonal will be mapped onto $\overline{CI}$ if each rotation is applied exactly once?

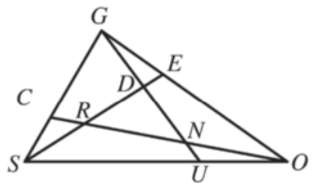
8 A puppy weighs 3/4 of its weight plus 3/4 lb. How many pounds does the puppy weigh?

9 In a group of 50 girls, each girl is either blonde or brunette, and each has either blue or brown eyes. Find the number of brown-eyed blondes if 14 girls are blue-eyed blondes, 31 are brunettes, and 18 have brown eyes.

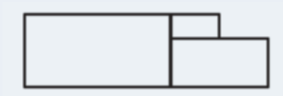
10 Compute the digit in the thousands place when calculating the following sum:

$$6 + 66 + 666 + 6666 + \cdots + 6{,}666{,}666{,}666$$

11 The points *C*, *E*, and *U* are the 1/3 points on the respective sides of $\triangle SGO$. Which triangle in the figure has exactly 1/7 of the area of $\triangle SGO$?

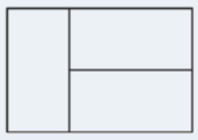


12 Three similar rectangles with integral widths and lengths in the ratio 1 : 2 form a nonconvex hexagon, as shown. The width of the largest rectangle is 3. Find the sum of the digits of the hexagon's perimeter.

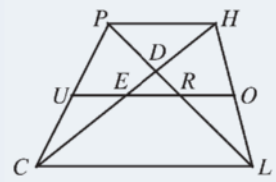


13 The equation $x^2 + px + q = 0$, $q \neq 0$, has two unequal roots such that the squares of the roots are the same as the two roots. Calculate the product $p \cdot q$.

14 A rectangular sheet of paper 2 in.× x in., $x > 2$, is cut into three congruent pieces by two straight cuts, as shown. Find the number of sq. in. in the area of the original rectangle.



15 In trapezoid *CPHL* with median *OU*, *CL* = 20 and *PH* = 12. Of the many segments in the figure, which one must have length 4?



16 Find all integral values of x such that

$$2^{2x^2-9x+4} = 5^{x^2-x-12}.$$

17 The repeating decimal 0.3333… in base 10 is the fraction 1/3. The same repeating decimal in base 9 can be written in reduced form as a/b where a and b are in base 10. Find the sum

$$a + b.$$

18 The expression

$$x^4 + 11x^3 + 30x^2 + 53x - 7$$

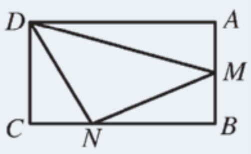
factors into two nonfactorable quadratics with integral coefficients. Find the sum of all the coefficients of the quadratics.

19 Find the value of k such that the graphs of the lines defined by the three equations

$$14x + 3y = 9,$$
$$9x - 2y = 49, \text{ and}$$
$$-7x - 6y = k$$

are concurrent.

20 The area of the rectangle shown is 48. The midpoint of $\overline{AB}$ is *M*, and *N* is 1/3 the distance from *C* to *B*. Compute the area of $\triangle DNM$.



21 In 1941 Fred's children were both between the ages of 10 and 20. The sum of the cube of one child's age and the square of the other child's age gives the year in which Fred's wife was born. How old was his wife in 1941?

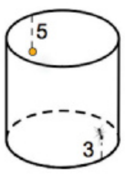
22 Compute the maximum value for the following function:

$$f(x) = 3\sin(3x) + 4\cos(3x)$$

23 The perimeter of a sector of a circle is 40. Find the radius of the sector that maximizes its area.

24 A teacher bought 100 puzzles for \$82.90 to give out to the members of her math team. She bought three different types of puzzle. Type A cost 40¢ each, type B cost 70¢ each, and type C cost \$1 each. How many more type C puzzles than type A puzzles did she buy?

25 An ant on the outside of a glass, 3 in. from the bottom, sees a drop of honey inside the glass, 5 in. from the top and exactly half way around the glass. The glass is 13 in. high and has a circumference of 16 in. Compute the length of the shortest path for the ant to reach the honey.



26 Given that the three quantities

$$\log_2(\log_3(\log_4(a))),$$
$$\log_3(\log_4(\log_2(b))), \text{ and}$$
$$\log_4(\log_2(\log_3(c)))$$

are all equal to 0, find

$$\frac{a}{b} + c.$$

27 The fraction

$$\frac{30}{\sqrt{3} + \sqrt{5} + \sqrt{8}}$$

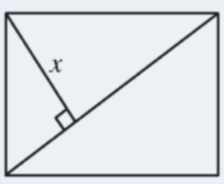
can be written as

$$p\sqrt{3} + q\sqrt{5} + r\sqrt{8} + s\sqrt{30}.$$

Find the value of the expression

$$p + q + r + s.$$

28 The rectangle shown in the diagram has width 15 and length 20. Find x, the length of the segment perpendicular to the diagonal.



29 In base b, the number 331_b is the square of an integer in base 10. Compute the smallest positive value of b.

30 A bicycle storeowner asks one of his employees to count the number of bicycles and tricycles in his store. The employee knows that the owner likes puzzles, so he tells him that there are a total of 169 wheels but only 152 pedals. How many more bicycles than tricycles are in the store?

31 Compute the positive geometric mean for the set of positive divisors of the number 324.

SOLUTIONS to calendar

Looking for more Calendar problems?
Visit www.nctm.org/publications/calendar/default.aspx?journal_id=2 for a collection of previously published problems—sortable by topic—from the *Mathematics Teacher.*

Edited by ***Margaret Coffey,*** *Margaret.Coffey@fcps.edu, Thomas Jefferson High School for Science and Technology, Alexandria, VA 22312, and* ***Art Kalish,*** *artkalish@verizon.net, Syosset High School, Syosset, NY 11791*

The Editorial Panel of the *Mathematics Teacher* is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly Calendar. Send problems to the Calendar editors. Remember to include a complete solution for each problem submitted.

1. MA. Consider the side labeled AN to be the front of the cube. The bottom could be the side labeled TA, the right would be RI, and the left would be NI. This leaves HU on the top and MA on the back.

2. $\overline{RT}$. From any point in the interior of a rectangle, the sum of the squares of the lengths of segments drawn from a pair of opposite corners will equal the sum of the squares of the lengths of the segments drawn from the remaining two vertices. (To prove this theorem, draw lines parallel to the sides of the rectangle through the selected interior point and apply the Pythagorean theorem several times. This theorem may be extended to include any point in the plane of the rectangle as well as any point in three-dimensional space.) In this case, $AT^2 + CT^2 = HT^2 + RT^2 \rightarrow AT^2 - HT^2 + CT^2 = RT^2$.

3. IN. The number 1 is assigned to the letters B, I, P, and W. The number 6 is assigned to the letters G, N, and U. The only word that can be made from these letters is IN.

4. 4. We can readily see that 1 satisfies both equations. There are at most two intersections of the graphs of a line and an exponential function; therefore, after testing a few more numbers, we might discover the number 3 as the other solution. These solutions may also be found by graphing the two functions and locating their intersections or by finding the zeros of the graph of $y = 12x - 9 - 3^x$. The sum of the abscissas of these solutions is 4.

5. 9. Let t be the tens digit and u be the units digit, and set up the equation $10t + u = 3tu$. Solving for t gives us $t = u/(3u - 10)$. Since t and u are single digits, $1 \le 3u - 10 \le 9 \rightarrow 4 \le u \le 6$. Testing the three possibilities yields solutions $u = 4$, $t = 2$, and $u = 5$, $t = 1$. Therefore, the numbers are 24 and 15, and the positive difference is 9.

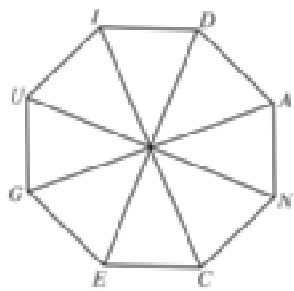
6. 2. Compute as follows:

$$\frac{{}_{2011}C_{1006}}{{}_{2010}C_{1005}} = \frac{\frac{2011!}{1006!\cdot 1005!}}{\frac{2010!}{1005!\cdot 1005!}}$$

$$= \frac{2011!}{1006!\cdot 1005!} \cdot \frac{1005!\cdot 1005!}{2010!}$$

$$= \frac{2011}{1006} \approx 2$$

7. $\overline{GA}$. Each angle at the center of the octagon is 45° since all the angles are congruent and must add to 360°. The three rotations in any order are equivalent to one rotation of 270° counterclockwise or 90° clockwise mapping diagonal CI onto diagonal GA.



8. 3. Let p be the weight of the puppy, so $3p/4 + 3/4 = p$. Solving for p gives $p = 3$.

***Alternate solution*:** Since 1/4 of the puppy's weight must be 3/4 lb., the puppy must weigh 4 • (3/4) = 3 lbs.

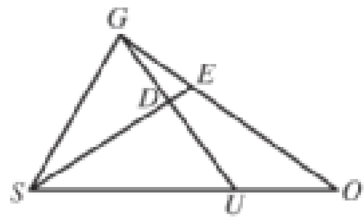
9. 5. One way to analyze the data is to create a chart such as this one. Fill in the initial data and then calculate each of the missing values. Since there are 50 girls and 31 are brunettes, there must be 19 blondes. Since 14 of the blondes are blue-eyed, 5 must be brown-eyed.

	Blonde	Brunette	Total
Blue	14		
Brown			18
Total		31	50

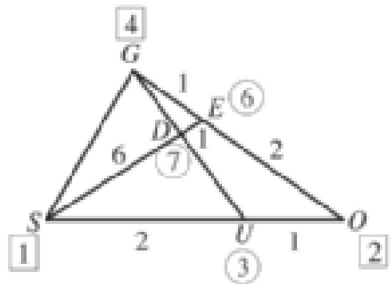
10. 7. The ten 6s in the units column have a sum of 60. Therefore, place a 0 in the units column of the sum and carry the 6. In the tens column, there are nine 6s plus the carried 6, so the sum is 60. Place a 0 in the tens column and carry the 6. In the hundreds column, there are eight 6s plus the carried 6, so the sum is 54. Place a 4 in the hundreds column of the sum and carry a 5. Finally, in the thousands column, there are seven 6s plus the carried 5, resulting in a sum of 47. Thus, the thousands column contains a 7 in the sum.

11. $\triangle RDN$. First, consider only two of the trisection points and their corresponding segments (cevians) (see **fig. a**). If the area of $\triangle SGO$ is n, then the area of $\triangle SGE = n/3$ since the two triangles have the same height and their bases are in a ratio of 1 : 3. Using mass-point geometry (see **fig. b** and note below), we can determine that the ratio $DE:SE$ is 1 : 7, so the area of $\triangle GED = n/21$, and the area of $\triangle SGD = 6n/21 = 2n/7$. Similarly, we can show that the area of $\triangle CRS = n/21$, so the area of quadrilateral $CGDR$ is $5n/21$. This result is the same for the other quadrilaterals $DEON$ and $SRNU$. The sum of the areas of the figures surrounding $\triangle RDN$ is $18n/21 = 6n/7$, leaving an area of $3n/21$ for $\triangle RDN$, which is $n/7$, or 1/7 the area of $\triangle SGO$ (see **fig. c**).

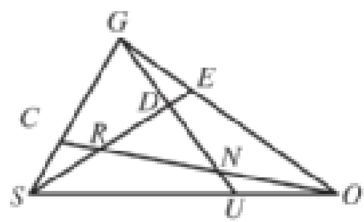
Note: Mass-point geometry allows us to use the idea of a lever and a fulcrum to find the weights at vertices on the basis of their distance to the fulcrum. The numbers on the sides of the triangle represent the relative lengths from vertex to fulcrum. The boxed values are the weights assigned at each vertex on the basis of the idea that balance is obtained when weight times distance is constant along an edge. The circled numbers are the total weight at the fulcrum. For example, since U is a fulcrum on side SO and $SU = 2UO$, the weight at S is half the weight at O. Arbitrarily assign 2 at O and 1 at S. Therefore, the total weight along $\overline{SO}$ is 3, placed at the fulcrum U. There are numerous references to mass point geometry or mass points on the Internet, and any search agent will locate them.



(a)



(b)



(c)

12. 8. Since the width of the largest rectangle is 3, its length must be 6. The side measures are all integers, so the width of the smallest rectangle is 1, and the width of the middle-size rectangle is 2. The rectangles' lengths are 2 and 4, respectively. Thus, the perimeter of the figure is 26, and the sum of the digits is 8.

13. 1. Let the roots of the quadratic be A and B. Then the roots are also A^2 and B^2. Since q is the product of the roots and p is the negative of the sum of the roots, we have $AB = A^2B^2$ and $A + B = A^2 + B^2$. From the first equation, we can see that $AB = 0$ or $AB = 1$. The problem states that $q \neq 0$, so $AB \neq 0$. Thus, $AB = 1$. In the second equation, add $2AB$ or 2 to both sides to get $A + B + 2AB = A^2 + 2AB + B^2 \rightarrow A + B + 2 = (A + B)^2$. This is a quadratic equation with the variable $A + B$. Set equal to 0 and solve: $(A + B)^2 - (A + B) - 2 = 0 \rightarrow ((A + B) - 2) \cdot ((A + B) + 1) = 0 \rightarrow A + B = 2$ or $A + B = -1$. If $A + B = 2$, the original equation is $x^2 - 2x + 1 = 0$, and the two roots are equal, violating one of the initial conditions. If $A + B = -1$, the equation is $x^2 + x + 1 = 0$, and all the conditions are met. Therefore, $pq = 1$.

***Alternate solution*:** Apply the quadratic formula to find that the roots are

$$x = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Since the squares of these two roots are the same as the roots themselves and neither root is zero, we have

$$\left(\frac{-p \pm \sqrt{p^2 - 4q}}{2}\right)^2$$

$$= \frac{p^2 \pm 2p\sqrt{p^2 - 4q} + p^2 - 4q}{4}$$

$$= \frac{p^2 - 2q \pm p\sqrt{p^2 - 4q}}{2},$$

which must be the same as

$$\frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

This implies that $p = \pm 1$ and that $p^2 - 2q = -p$. If $p = -1$, then $q = 0$, which cannot be, so p must be 1, which leaves $q = 1$ and $p \cdot q = 1$. We leave it to readers to confirm that the squares of the roots of $x^2 + x + 1 = 0$ are the same as the roots.

14. 6. Since the three small rectangles are congruent, the dimensions of each must be 1×2. Thus, $x = 3$, and the area of the original rectangle is 6.

15. $\overline{ER}$. The length of the median of a trapezoid is the average of the bases, so $UO = (20 + 12)/2 = 16$. Since the median is parallel to the two bases and three parallel lines divide all transversals proportionately, E and R are midpoints of $\overline{CH}$ and $\overline{LP}$, respectively. Since a line segment that bisects two sides of a triangle is half the length of the third side, $UE = RO = PH/2 = 6$. Finally, $ER = UO - UE - RO = 16 - 6 - 6 = 4$. (Note that PC or HL or either of their parts may equal 4, but none must be 4.)

16. 4. The only time an integral power of 2 can equal an integral power of 5 occurs when both powers are 0. We need to find the common integral solution to the equations $2x^2 - 9x + 4 = 0$ and $x^2 - x - 12 = 0$: $2x^2 - 9x + 4 = 0 \to (2x - 1)(x - 4) = 0 \to x = 1/2$ or 4, and $x^2 - x - 12 = 0 \to (x + 3)(x - 4) = 0 \to x = -3$ or 4. Therefore, the common solution is 4.

17. 11. Repeating decimals can be converted into fractions in the following way. In base 10, let $n = 0.333\ldots$, so $10n = 3.333\ldots$. Subtracting the two equations, we get $9n = 3$, so $n = 3/9$, which is reducible to 1/3. In base 9, we let $n = 0.333\ldots$, and then, since the place values are powers of 9, we obtain $9n = 3.333\ldots$ (the 9 is in base 10), so $8n = 3$ and $n = 3/8$. Finally, $3 + 8 = 11$.

***Alternate solution*:** In base 9, the notation 0.333… indicates the following series:

$$3\cdot 9^{-1} + 3\cdot 9^{-2} + 3\cdot 9^{-3} + \cdots = \frac{3}{9} + \frac{3}{9^2} + \frac{3}{9^3} + \cdots$$

This is an infinite geometric series with a common ratio of 1/9. The sum of this series is

$$S = \frac{a}{1-r} = \frac{\frac{3}{9}}{1-\frac{1}{9}} = \frac{3}{8}.$$

Then $3 + 8 = 11$.

18. 19. The trinomial factors of $x^4 + 11x^3 + 30x^2 + 53x - 7$ are $x^2 + ax + b$ and $x^2 + cx + d$, where $b \cdot d = -7$, limiting the choices for b and d. We can begin with either $(x^2 + ax - 7)(x^2 + cx + 1)$ or $(x^2 + ax + 7)(x^2 + cx - 1)$. Since most of the coefficients in the product are positive, a good first guess would be to make the 7 positive, as in the second case. We know from the cubed term that $a + c = 11$ and from the linear term that $7c - a = 53$. Adding these last two equations yields $8c = 64 \to c = 8$ and $a = 3$. Thus, the factors are $(x^2 + 3x + 7)$ and $(x^2 + 8x - 1)$, and the sum of the coefficients is $1 + 3 + 7 + 1 + 8 - 1 = 19$.

19. 45. Since the three lines must coincide, the third equation contains the point of intersection of the first two equations. Solving the first two simultaneously yields the coordinates $(3, -11)$. Substitute these coordinates into the third equation to get $-7(3) - 6(-11) = k$, so $k = 45$.

20. 20. Let $CD = 2x$ and $AD = 3y$. Then the area of $\triangle DNM = 6xy$ – the areas of the three surrounding right triangles $= 6xy - (xy + xy + 3xy/2) = 5xy/2$. We know that $6xy = 48 \to xy = 8$. Therefore, $5xy/2 = 20$.

Alternate solution: We can partition the rectangle into smaller rectangles such that the areas of the three triangles that surround $\triangle DNM$ are easily seen as fractions of rectangle $ABCD$'s area. This approach eliminates the need for variables. Through M, the midpoint of $\overline{AB}$, draw a line parallel to $\overline{AD}$ that intersects $\overline{DC}$ at P. The area of rectangle $AMPD$ is 1/2 the area of rectangle $ABCD$, so the area of $\triangle AMD = 48/4 = 12$ sq. units. Next, draw a line through N parallel to $\overline{DC}$ that intersects $\overline{AD}$ at Q. Since N is 1/3 the distance from C to B, the area of rectangle $CDQN$ is 1/3 the area of rectangle $ABCD$. Therefore, the area of $\triangle NDC = 48/6 = 8$. The area of rectangle $ABNQ$ is 2/3 the area of rectangle $ABCD$. Let R be the intersection of the two newly drawn lines, so the area of rectangle $MBNR$ is 1/3 the area of rectangle $ABCD$, and the area of $\triangle MBN = 48/6 = 8$. Finally, the area of $\triangle DNM = 48 - 12 - 8 - 8 = 20$, as above. We note that 20/48 is 5/12, the area of rectangle $ABCD$.

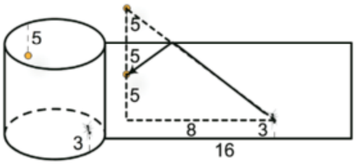
21. 44. By examining the sum of the cube and the square of integers between 10 and 20, we see that the only possible year that Fred's wife could have been born would be $1897 = 12^3 + 13^2$. Thus, in 1941 she was 44 years old. (The nearest totals to 1897 are $12^2 + 12^3 = 1872$ and $14^2 + 12^3 = 1924$, neither of which is a reasonable response to the question.)

22. 5. Apply the formula $\sin(A + B) = \sin A \cdot \cos B + \sin B \cdot \cos A$. Then $k\sin(3x + B) = k\sin(3x)\cdot\cos B + k\sin B \cdot\cos(3x)$. It follows that $k\cos B = 3$ and $k\sin B = 4$. Squaring both sides of the previous two equations and adding them results in $k^2(\cos^2 B + \sin^2 B) = 9 + 16 = 25$. But $\cos^2 B + \sin^2 B = 1$, so $k^2 = 25$ and $k = \pm 5$. In either case, the maximum value for the function $f(x)$ will be 5.

23. 10. A sector of a circle is the portion of a circle enclosed by two radii and an arc. The arc length $s = \theta\cdot r$, its perimeter $P = 2r + s$, and the area $A = \theta\cdot r^2/2$. Since $P = 40$, $\theta = (40 - 2r)/r$. Substituting this value into the area formula and simplifying results in $A(r) = 20r - r^2$. The maximum value for a quadratic function occurs along the axis of symmetry, so when $r = -20/-2 = 10$, the area is maximum.

24. 43. Let A, B, and C be the number of each type of puzzle. Then $A + B + C = 100$, and, in terms of cents, $40A + 70B + 100C = 8290$. Multiply the first equation by 70 and subtract the result from the second equation. The result is $-30A + 30C = 1290$. Therefore, $C - A = 43$.

25. 17 in. The ant must choose to move right or left, and then it must head for a point on the rim of the glass. To find that point, imagine unrolling the glass to form a rectangle. Place the ant 3 in. from the bottom; then, since the ant and the honey are on opposite sides of the glass, reflect the honey over the top of the glass. Form the right triangle shown in the diagram. Since the height of the glass is 13 in., the height of the triangle is $13 + 5 - 3 = 15$. The base of the triangle is half the circumference of the glass, or 8 in., since the honey is directly opposite the ant. Recognize the Pythagorean triple 8-15-17 or apply the Pythagorean theorem. The ant's actual path is displayed as a solid line (see diagram).



26. 13. Since

$$\log_2(\log_3(\log_4(a))) = \log_3(\log_4(\log_2(b))) = \log_4(\log_2(\log_3(c))) = 0,$$

we know that

$$\log_3(\log_4(a)) = 2^0 = 1 \to \log_4(a) = 3 \to a = 4^3 = 64,$$

$$\log_4(\log_2(b)) = 3^0 = 1 \to \log_2(b) = 4 \to b = 2^4 = 16,$$

and

$$\log_2(\log_3(c)) = 4^0 = 1 \to \log_3(c) = 2 \to c = 3^2 = 9.$$

Thus, $a/b + c = 64/16 + 9 = 4 + 9 = 13$.

27. 6. Multiply the numerator and the denominator by one of the conjugates of the denominator. For example:

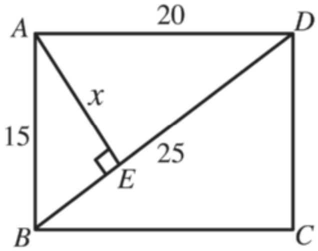
$$\frac{30}{\sqrt{3}+\sqrt{5}+\sqrt{8}} \cdot \frac{\left(\sqrt{3}+\sqrt{5}\right)-\sqrt{8}}{\left(\sqrt{3}+\sqrt{5}\right)-\sqrt{8}} = \frac{30\left(\sqrt{3}+\sqrt{5}-\sqrt{8}\right)}{\left(\sqrt{3}+\sqrt{5}\right)^2-\left(\sqrt{8}\right)^2} = \frac{30\left(\sqrt{3}+\sqrt{5}-\sqrt{8}\right)}{3+5+2\sqrt{15}-8} = \frac{30\left(\sqrt{3}+\sqrt{5}-\sqrt{8}\right)}{2\sqrt{15}}$$

Rationalizing the denominator results in the following:

$$\frac{15\left(\sqrt{3}+\sqrt{5}-\sqrt{8}\right)}{\sqrt{15}} \cdot \frac{\sqrt{15}}{\sqrt{15}} = \frac{15\sqrt{15}\left(\sqrt{3}+\sqrt{5}-\sqrt{8}\right)}{15} = \sqrt{15}\left(\sqrt{3}+\sqrt{5}-\sqrt{8}\right) = 3\sqrt{5}+5\sqrt{3}-2\sqrt{30}$$

The sum of the coefficients is $3 + 5 + 0 - 2 = 6$.

28. 12. We know that $\triangle ABE \sim \triangle BDC$ since $m\angle ABE = m\angle BDC$ and both contain a right angle. Applying a multiple of the 3-4-5 Pythagorean triple or using the Pythagorean theorem, we find that $BD = 25$. Therefore, all we need do is set up and solve the proportion $x/15 = 20/25 \to x = 12$.



Alternate solution 1: Apply the Pythagorean theorem to find BD, the length of the hypotenuse of $\triangle BAD$. The product of a base and corresponding height for any triangle is constant. In $\triangle BAD$, $AD \cdot AB = BD \cdot AE \to 20 \cdot 15 = 25x \to x = 12$.

Alternate solution 2: Use the theorems dealing with the altitude drawn to the hypotenuse of a right triangle first to express AB as the mean proportional between BE and BD. Then use that result to express x as the mean proportional between BE and ED.

29. 7. The number 331_b converts into base 10 using the expression $3b^2 + 3b + 1$, and this number must be a perfect square. Hence, for some integer n, $3b^2 + 3b + 1 = n^2$, and $b > 3$. Apply the quadratic formula to the equation $3b^2 + 3b + 1 - n^2 = 0$ to get the following:

$$b = \frac{-3 \pm \sqrt{9-12\left(1-n^2\right)}}{6} = \frac{-3+\sqrt{12n^2-3}}{6}$$

We reject the negative value since b must be positive. The least values of n that make b a positive integer are ±1 and ±13. If $n = \pm 1$, then $b = 0$, which cannot be, and if $n = \pm 13$, then $b = 7$. (The next larger value of n that results in an integral value for b is 181, forcing b to be 104.) Trial and success might be a much faster way to arrive at the result.

30. 42. The difference between the number of wheels and the number of pedals yields the number of tricycles. Thus, there are 17 tricycles. These require 34 pedals, so the number of pedals remaining, $152 - 34 = 118$, determines that there are 59 bicycles. Thus, there are $59 - 17 = 42$ more bicycles than tricycles.

31. 18. Divisors can be paired so that we do not need to list all the divisors to realize that the answer is simply $\sqrt{324}$. Consider the following pairings: $1 \cdot 324$, $2 \cdot 162$, $3 \cdot 108$, and so on. The positive geometric mean of a set of n numbers is the positive nth root of their product. In this application, the square root of each product pair is 18, so the nth root of the n products is 18.

Alternate solution: Since 324 is a perfect square, it has an odd number of divisors. We have 7 pairs (1 and 324, 2 and 162, 3 and 108, and so on) and the unpaired square root—namely, 18. The product of these 15 divisors is $324^7 \cdot 18 = (18^2)^7 \cdot 18 = 18^{15}$. The 15th root of 18^{15} is 18.

Answer key

	MA (1)	RT (2)	IN (3)
4 (4)	9 (5)	2 (6)	GA (7)
3 (8)	5 (9)	7 (10)	RDN (11)
8 (12)	1 (13)	6 (14)	ER (15)
4 (16)	11 (17)	19 (18)	45 (19)
20 (20)	44 (21)	5 (22)	10 (23)
43 (24)	17 (25)	13 (26)	6 (27)
12 (28)	7 (29)	42 (30)	18 (31)