MA4140: Abstract Algebra (Fall 2011)
Final Exam

Your Name:

Names of any collaborators:

Instructions

This exam is worth a total of 86 points and 15% of your overall grade. For each part of the exam, read the instructions carefully.

I expect your proofs to be *well-written, neat, and organized.* You should write in *complete sentences.* Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts. Feel free to type up your final version.

The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem 28, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not covered.
3. You are NOT allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are NOT allowed to copy someone else's work.
5. You are NOT allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

The exam is due to my office by 5PM on **Friday, December 16**. You should turn in this cover page and all of the work that you have decided to submit.

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

Written by D.C. Ernst

Part 1

Complete each of the following problems. You should provide sufficient justification where necessary.

1. (5 points) Suppose that f is a homomorphism from $\mathbb{Z}_{30}$ to $\mathbb{Z}_{30}$ such that $K_f = \{0, 10, 20\}$. If $f(23) = 6$, determine all elements that map to 6.

2. (3 points each) Suppose $\phi : D_3 \to \mathbb{Z}_2$ is a group homomorphism satisfying

$$\phi(e) = 0, \phi(r) = 0, \phi(r^2) = 0, \phi(s) = 1, \phi(sr^2) = 1.$$

   (a) Assuming that ϕ is a group homomorphism, find $\phi(sr)$.
   (b) Find $\ker(\phi)$.
   (c) Explain why ϕ is *not* an isomorphism.
   (d) What well-known group is $D_3/\ker(\phi)$ isomorphic to? Explain your answer.

3. (5 points) Consider the group $Q_4 = \{\pm 1, \pm i, \pm j, \pm k\}$ and let $H = \langle -1 \rangle$.* It turns out that H is normal in Q_4. (You do *not* need to prove this.) What well-known group is Q_4/H isomorphic? Be sure to justify your answer.

4. (3 points each) Consider $D_4 = \langle s, r : s^2 = r^4 = e, sr = r^3 s\rangle$. Let $H_1 = \{e, s\}$ and $H_2 = \{e, s, r^2, sr^2\}$. It turns out that both H_1 and H_2 are subgroups of D_4. (You do *not* need to prove this.)

   (a) Briefly justify why H_2 is a normal subgroup of D_4.
   (b) Briefly justify why H_1 is a normal subgroup of H_2.
   (c) Show that H_1 is not a normal subgroup of D_4.†
   (d) Briefly explain why D_4/H_2 and H_2/H_1 are isomorphic.

5. (5 points) Briefly explain why Q_4 and D_4 are not isomorphic.

Part 2

(15 points) For each $n \in \{1, 2, \ldots, 15\}$, make as complete a list as possible which gives all of the non-isomorphic n-element groups. In each case, you should briefly justify your answer by citing relevant theorems (including ones that I discussed in class that may not be in the course notes). You do *not* need to list groups that are isomorphic. For example, we know that K_4 and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are isomorphic, so when handling $n = 4$, you only need to list one of these. In some situations (e.g., $n = 8$), you may not be able to list all of the possible groups of a given order or you may not be able to prove that your list is complete. In these situations, you should state that this is the case. You may use Group Explorer to help you, but you should not rely on it for justification.‡

Part 3

(8 points each) Prove any **four** of the following theorems.

Theorem 1. Let G_1 and G_2 be groups and let H_1 and H_2 be normal subgroups of G_1 and G_2, respectively. Then $H_1 \times H_2$ is a normal subgroup of $G_1 \times G_2$.

*You may consult Group Explorer if you need to be reminded how to multiply in this group.
†This shows that being a normal subgroup is not transitive.
‡In class, I mentioned that I wanted you to do this for 1–20, but you only need to handle 1–15.

Theorem 2. Let G be a group of order p^2 such that p is prime and let $H \leq G$ such that the order of H is p. Then H is normal in G.

Theorem 3. Let G be a group. If $G/Z(G)$ is cyclic, then G is abelian.

Theorem 4. Let $f : G \to G'$ be a group homomorphism. Then K_f is a normal subgroup of G.[§]

Theorem 5. If N is a normal subgroup of G, then the natural homomorphism $h : G \to G/N$, given by $h(x) := Nx$, is a homomorphism from G onto G/N.[¶]

Theorem 6. Let G be a group and let $H \leq G$. Then H is a normal subgroup of the normalizer of H in G.[‖]

Theorem 7. Let $n, m \in \mathbb{Z}$. Then $(\mathbb{Z} \times \mathbb{Z})/(n\mathbb{Z} \times m\mathbb{Z}) \cong \mathbb{Z}_n \times \mathbb{Z}_m$.[**]

Theorem 8. Let N be a normal subgroup of a group G. If $g \in G$, then the order of Ng (in G/N) divides the order of g (in G).

By the way, the upshot of Theorem 4 and Theorem 5 is that normal subgroups and kernels are really the same thing.

[§] This is Corollary 90 of our course notes.

[¶] This is Theorem 95 from our course notes. Note that you need to prove that h is a homomorphism and that it is onto.

[‖] Recall that the normalizer of H in G is defined to be $N_G(H) = \{g \in G : g^{-1}hg \in H \text{ for all } h \in H\}$. The normalizer is a pretty good name for this set, huh?

[**] Since $\mathbb{Z}$ is abelian, $\mathbb{Z} \times \mathbb{Z}$ is abelian. This implies that all subgroups of $\mathbb{Z} \times \mathbb{Z}$ are normal. In particular, $n\mathbb{Z} \times m\mathbb{Z} \trianglelefteq \mathbb{Z} \times \mathbb{Z}$, so that $(\mathbb{Z} \times \mathbb{Z})/(n\mathbb{Z} \times m\mathbb{Z})$ is a well-defined group. So, all you need to do is exhibit an isomorphism and prove that it is, in fact, an isomorphism.

