1 Introduction to Mathematics

1.3 Negating Implications and Proof by Contradiction

So far we have discussed how to negate propositions of the form A, $A \wedge B$, and $A \vee B$ for propositions A and B. However, we have yet to discuss how to negate propositions of the form $A \implies B$.

Theorem 1.43. Let A and B be propositions. Conjecture an equivalent way of expressing the conditional proposition $A \implies B$ as a proposition involving the disjunction symbol $\vee$ and possibly the negation symbol $\neg$, but not the implication symbol $\implies$. Prove your conjecture using a truth table.

Exercise 1.44. Let A and B be the propositions "Darth Vader is a hippie" and "Sarah Palin is a liberal", respectively. Using Theorem 1.43, express $A \implies B$ as an English sentence involving the disjunction "or."

Theorem 1.45 (*)**.** Let A and B be two propositions. Conjecture an equivalent way of expressing the proposition $\neg(A \implies B)$ as a proposition involving the conjunction symbol $\wedge$ and possibly the negation symbol $\neg$, but not the implication symbol $\implies$. Prove your conjecture using previous results.

Exercise 1.46. Let A and B be the propositions in Exercise 1.44. Using Theorem 1.45, express $\neg(A \implies B)$ as an English sentence involving the conjunction "and."

Exercise 1.47. The following proposition is *false*. Negate this proposition to obtain a true statement. Write your statement as a conjunction.

If $.\overline{99} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$, then $.\overline{99} \neq 1$.

You do *not* need to prove your new statement.

Recall that a proposition is exclusively either true or false. That is, a proposition can never be both true and false. This idea leads us to the next definition.

Definition 1.48. A compound proposition that is always false is called a **contradiction**. A compound statement that is always true is called a **tautology**.

Theorem 1.49. Let A be a proposition. Then $\neg A \wedge A$ is a contradiction.

Exercise 1.50. Provide an example of a tautology.

Suppose that we want to prove some proposition P (which might be something like $A \implies B$ or possibly more complicated). One approach, called **proof by contradiction**, involves assuming $\neg P$ and then logically deducing a contradiction of the form $Q \wedge \neg Q$, where Q is some proposition (possibly equal to P). Since this is absurd, it cannot be the case that $\neg P$ is true, which implies that P is true.

Among other situations, proof by contradiction can be useful for proving statements of the form $A \implies B$, where B is worded negatively or $\neg B$ is easier to "get your hands on."

Question 1.51. Let A and B be propositions. Describe a general strategy for proving $A \implies B$ via proof by contradiction.

Prove the following theorem in two ways: (i) prove the contrapositive and (ii) use proof by contradiction.

This work is an adaptation of notes written by Stan Yoshinobu of Cal Poly and Matthew Jones of California State University, Dominguez Hills.

Problem 1.52 (*)**.** Assume that $x \in \mathbb{Z}$. If x is odd, then 2 does not divide x.

Prove the following theorem by contradiction.

Theorem 1.53 (*)**.** Assume that $x, y \in \mathbb{N}$. If x divides y, then $x \leq y$.

Question 1.54. What obstacles (if any) are there to proving the previous theorem directly without using proof by contradiction.