2 Set Theory and Topology

2.1 Sets

At its essence, all of mathematics is built on set theory.

Definition 2.1. A **set** is a collection of objects called **elements**. If A is a set and x is an element of A, we write $x \in A$. Otherwise, we write $x \notin A$.

Definition 2.2. The set containing no elements is called the **empty set**, and is denoted by the symbol $\emptyset$.

Definition 2.3. The language associated to sets is specific.

$$S = \{x \in A : x \text{ satisfies some condition}\}$$

The first part "$x \in A$" denotes what type of x is being considered. The statements to the right of the colon are the conditions that x must satisfy, in order to be members of the set. This notation is read as "The set of all x in A such that x satisfies some condition," where "some condition" is something specific about the restrictions on x relative to A. This type of notation is often called **set builder notation**.

Exercise 2.4. Unpack each of the following sets into a description using a sentence and see if you can determine exactly what elements each set contains.

1. $M = \{x \in \mathbb{R} : x \geq 2\}$
2. $A = \{x \in \mathbb{N} : x = 3k \text{ for some } k \in \mathbb{N}\}$
3. $T = \{t \in \mathbb{R} : t^2 \leq 2\}$
4. $H = \{t \in \mathbb{R} : t = 1 - \frac{1}{n}, \text{ where } n \in \mathbb{Z}\}$

Exercise 2.5. Write each of the following sentences using set builder notation.

1. Suppose R is the set of all real numbers x such that x is less than $-\sqrt{2}$.
2. Suppose A is the set of all real numbers y, such that y is greater than -12 and less than 42.4.
3. Suppose D is the set of all even natural numbers.

Definition 2.6. If A and B are sets, then we say that A is a **subset** of B, written $A \subseteq B$, if every element of A is also an element of B.

Remark 2.7. Observe that $A \subseteq B$ is equivalent to "For all x (in the universe of discourse), if $x \in A$, then $x \in B$." Since we know how to deal with "for all" statements and conditional propositions, we know how to go about proving $A \subseteq B$.

Question 2.8. Suppose that A and B are sets. Describe a general strategy for proving that $A \subseteq B$.

Theorem 2.9. Let S be a set. Then

1. $S \subseteq S$,
2. $\emptyset \subseteq S$.

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Exercise 2.10. List all of the subsets of $A = \{1, 2, 3\}$. Any conjectures about how many there might be for a set with n elements?

Theorem 2.11 (*, Transitivity of subsets)**.** Suppose that A, B, and C are sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Definition 2.12. If $A \subseteq B$, then A is called a **proper subset** provided that $A \neq B$. In this case, we may write $A \subset B$ or $A \subsetneq B$.*

Definition 2.13 (Interval Notation)**.** For $a, b \in \mathbb{R}$ with $a < b$, we define the following.

1. $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
2. $(a, \infty) = \{x \in \mathbb{R} : a < x\}$
3. $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$
4. $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

We analogously define $[a, b)$, $(a, b]$, $[a, \infty)$, and $(-\infty, b]$.

Exercise 2.14. Provide two examples of proper subsets of $[0, 1]$.

Here are some more definitions. In each case, take U to be the universe of discourse.

Definition 2.15. The **union** of the sets A and B is $A \cup B = \{x \in U : x \in A \text{ or } x \in B\}$

Definition 2.16. The **intersection** of the sets A and B is $A \cap B = \{x \in U : x \in A \text{ and } x \in B\}$

Definition 2.17. The **set difference** of the sets A and B is $A \setminus B = \{x \in U : x \in A \text{ and } x \notin B\}$

Definition 2.18. The **complement of** A (relative to U) is the set $A^c = U \setminus A = \{x \in U : x \notin A\}$

Definition 2.19. If two sets A and B have the property that $A \cap B = \emptyset$, then we say that A and B are **disjoint** sets.

Exercise 2.20. Suppose that the universe of discourse is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 5\}$, and $C = \{2, 4, 6, 8\}$. Find each of the following.

1. $A \cap C$
2. $A \cap B$
3. $A \cup C$
4. $A \cup B$
5. $A \setminus B$
6. $B \setminus A$
7. $C \setminus B$
8. $B \cap C$
9. B^c

* *Warning:* Some books use $\subset$ to mean $\subseteq$.

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10. A^c

11. $(A \cup B)^c$

12. $A^c \cap B^c$

Exercise 2.21. Suppose that the universe of discourse is $U = \mathbb{R}$. Let $A = [-3, -1)$ and $B = (-2.5, 2)$, $C = (-2, 0]$. Find each of the following.

1. A^c
2. $A \cap C$
3. $A \cap B$
4. $A \cup C$
5. $A \cup B$
6. $(A \cap B)^c$
7. $(A \cup B)^c$
8. $A \setminus B$
9. $A \setminus (B \cup C)$
10. $B \setminus A$
11. $B \cap C$

Theorem 2.22 (*)**.** Let A and B be sets. If $A \subseteq B$, then $B^c \subseteq A^c$.

Definition 2.23. Two sets A and B are **equal** if and only if $A \subseteq B$ and $B \subseteq A$. In this case we write $A = B$.

Remark 2.24. Given two sets A and B, if we want to prove $A = B$, then we have to do two separate "mini" proofs: one for $A \subseteq B$ and one for $B \subseteq A$.

Theorem 2.25 (*)**.** Let A and B be sets. Then $A \setminus B = A \cap B^c$.

Theorem 2.26 (*, DeMorgan's Law)**.** Let A and B be sets. Then

1. $(A \cup B)^c = A^c \cap B^c$,
2. $(A \cap B)^c = A^c \cup B^c$.

(You only need to prove one of these; the other is similar.)

Theorem 2.27 (*, Distribution of Union and Intersection)**.** Let A, B, and C be sets. Then

1. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(You only need to prove one of these; the other is similar.)

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