3 Relations and Functions

3.5 Compositions and Inverses

Definition 3.91. If $f : X \to Y$ and $g : Y \to Z$ are functions, then a new function $g \circ f : X \to Z$ can be defined by $(g \circ f)(x) = g(f(x))$ for all $x \in \mathrm{Dom}(f)$.

Remark 3.92. It is important to notice that the function on the right is the one that "goes first."

Exercise 3.93. In each case, give examples of finite sets X, Y, and Z, and functions $f : X \to Y$ and $g : Y \to Z$ that satisfy the given conditions. Drawing bubble diagrams is sufficient.

1. f is onto, but $g \circ f$ is not onto.
2. g is onto, but $g \circ f$ is not onto.
3. f is one-to-one, but $g \circ f$ is not one-to-one.
4. g is one-to-one, but $g \circ f$ is not.

Theorem 3.94 (*)**.** If $f : X \to Y$ and $g : Y \to Z$ are both functions that are onto, then $g \circ f$ is also onto.

Theorem 3.95 (*)**.** If $f : X \to Y$ and $g : Y \to Z$ are both functions that are one-to-one, then $g \circ f$ is also one-to-one.

Corollary 3.96. If $f : X \to Y$ and $g : Y \to Z$ are both one-to-one correspondences, then $g \circ f$ is also a one-to-one correspondence.

Problem 3.97. Assume that $f : X \to Y$ and $g : Y \to Z$ are both functions. For each of the following statements, if the statement is true, then prove it. If the statement is false, provide a counterexample.

1. If $g \circ f$ is one-to-one, then f is one-to-one.
2. If $g \circ f$ is one-to-one, then g is one-to-one.
3. If $g \circ f$ is onto, then f is onto.
4. If $g \circ f$ is onto, then g is onto.

Definition 3.98. Let $f : X \to Y$ be a function. The relation f^{-1}, called f **inverse**, is defined via

$$f^{-1} = \{(f(x), x) : x \in X\}.$$

Remark 3.99. Notice that we called f^{-1} a relation and not a function. In some circumstances f^{-1} will be a function and sometimes it won't be.

Exercise 3.100. Provide an example of a function $f : X \to Y$ such that f^{-1} is *not* a function. A bubble diagram is sufficient.

Exercise 3.101. Provide an example of a function $f : X \to Y$ such that f^{-1} is a function. A bubble diagram is sufficient.

Theorem 3.102 (*)**.** Let $f : X \to Y$ be a function. Then f^{-1} is a function iff f is ________________.

This work is an adaptation of notes written by Stan Yoshinobu of Cal Poly and Matthew Jones of California State University, Dominguez Hills.

Theorem 3.103 (*)**.** Let $f : X \to Y$ be a function and suppose that f^{-1} is a function. Then

1. $(f \circ f^{-1})(x) = x$ for all $x \in Y$, and
2. $(f^{-1} \circ f)(x) = x$ for all $x \in X$.

(You only need to prove one of these statements; the other is similar.)

Theorem 3.104 (*)**.** Let $f : X \to Y$ and $g : Y \to X$ be functions such that f is a one-to-one correspondence. If $(f \circ g)(x) = x$ for all $x \in Y$ and $(g \circ f)(x) = x$ for all $x \in X$, then $g = f^{-1}$.

Remark 3.105. The upshot of the previous two theorems is that if f^{-1} is a function, then it is the only one satisfying the two-sided "undoing" property exhibited in Theorem 3.103.

The next theorem can be considered to be a converse of Theorem 3.104.

Theorem 3.106 (*)**.** Let $f : X \to Y$ and $g : Y \to X$ be functions satisfying $(f \circ g)(x) = x$ for all $x \in Y$ and $(g \circ f)(x) = x$ for all $x \in X$. Then f is a one-to-one correspondence.

Theorem 3.107 (*)**.** Let $f : X \to Y$ and $g : Y \to Z$ be functions. If f and g are both one-to-one correspondences, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.