1 Introduction to Mathematics (Continued)

1.5 More on Quantification

In the last section, we introduced the universal quantifier "for all" and the existential quantifier "there exists... such that." Here are a couple of important points to remember about quantification:

1. In order to have a proposition, all variables must be bound. That is, all variables must be quantified. This can happen in at least two ways:
   (a) The variables are explicitly bound by quantifiers in the same sentence,
   (b) The variables are implicitly bound by preceding sentences and/or by context. *Note:* Statements of the form "Let $x = \ldots$" and "Let $x \in \ldots$" bind the variable x and remove ambiguity.
2. The order of the quantification is important. Reversing the order of the quantifiers can substantially change the meaning of a proposition.

Using our logical connectives ("and", "or", "If..., then...", and "not") together with quantification, we can build very complex mathematical statements.

Example 1.66. Let f be a function and consider the formal definition of the limit of $f(x)$ as x approaches c:

For all $\epsilon > 0$, there exists $\delta > 0$ such that for all x, if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Exercise 1.67. Identify all the quantifiers from Example 1.66 and any logical connectives. Are there any implicit bound variables?

In order to study the abstract nature of complicated mathematical statements, it is useful to adopt some notation.

Definition 1.68. We use the symbol $\forall$ to denote the universal quantifier "for all" and the symbol $\exists$ to denote the existential quantifier "there exists... such that".

By using our abbreviations for the logical connectives and quantifiers, we symbolically represent mathematical propositions.

Example 1.69. For each of the following suppose our universe of discourse is the set of real numbers.

1. Consider the following (true) proposition:

   There exists x such that $x^2 - 1 = 0$.

   This proposition can be denoted symbolically as $(\exists x)(x^2 - 1 = 0)$.

2. Consider the following (false) proposition:

   For all $x \in \mathbb{N}$, there exists $y \in \mathbb{N}$ such that $y < x$.

   They can be represented symbolically as $(\forall x)(x \in \mathbb{N} \implies (\exists y)(y \in \mathbb{N} \implies y < x))$ or more simply as $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(y < x)$.

This work is an adaptation of notes written by Stan Yoshinobu of Cal Poly and Matthew Jones of California State University, Dominguez Hills.

3. Consider the following (true) proposition:

   Every positive real number has a multiplicative inverse.

   There are several ways of representing this statement symbolically. However, if you unpack what a multiplicative inverse is, you'll get something like $(\forall x)(x > 0 \implies (\exists y)(xy = 1))$. Alternatively, you can shorten the statement to $(\forall x > 0)(\exists y)(xy = 1)$.

Exercise 1.70. Convert the following statements into statements using only logical symbols. Assume that the universe of discourse is the set of real numbers.

1. There exists a number x such that $x^2 + 1$ is greater than zero.
2. There exists a natural number n such that $n^2 = 36$.
3. For every real number x, x^2 is greater than or equal to zero.

Exercise 1.71. Convert the statement in Example 1.66 into a statement using only logical symbols.

Remark 1.72. If $A(x)$ and $B(x)$ are predicates, then it is standard practice for the statement $A(x) \implies B(x)$ to mean $(\forall x)(A(x) \implies B(x))$ (where the universe of discourse for x needs to be made clear). In this case, we say that the universal quantifier is implicit.

Exercise 1.73. Find at least two examples earlier in the notes that exhibit the claim made in Remark 1.72. Attempt to write the statements symbolically.

Exercise 1.74. Convert the following proposition into a statement using only logical symbols. The universe of discourse is the set of real numbers. (Watch out for implicit quantifiers.)

If $\epsilon > 0$, then there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$.

Exercise 1.75. Unpack the following symbolic propositions into words:

1. $(\forall n \in \mathbb{N})(n^2 + 1 \neq 0)$
2. $(\exists N \in \mathbb{N})(\forall n > N)(\frac{1}{n} < 0.01)$

Remark 1.76. The symbolic expression $(\forall x)(\forall y)$ can be replaced with the simpler expression $(\forall x, y)$ as long as x and y are coming from the same set.

Exercise 1.77. For each of the following statements, (i) unpack the statement into words and (ii) determine whether the statement is true or false.

1. $(\forall n \in \mathbb{N})(n^2 \geq 5)$
2. $(\exists n \in \mathbb{N})(n^2 - 1 = 0)$
3. $(\forall m, n \in \mathbb{Z})(2|m \wedge 2|n \implies 2|(m + n))$
4. $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(x - 2y = 0)$
5. $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})(y \leq x)$

To whet your appetite for the next section, tackle the following questions.

Question 1.78. How would you go about proving a true statement of the form "For all x..."?

Question 1.79. If a statement is false, then its negation is true. How would you go about negating a statement involving quantifiers? In particular, what are the negations of $\forall$ and $\exists$, respectively?