# 2 Set Theory and Topology

At its essence, all of mathematics is built on set theory. In this chapter, we will introduce some of the basics of sets and their properties.

## 2.1 Sets

**Definition 2.1.** A **set** is a collection of objects called **elements**. If $A$ is a set and $x$ is an element of $A$, we write $x \in A$. Otherwise, we write $x \notin A$.

**Definition 2.2.** The set containing no elements is called the **empty set**, and is denoted by the symbol $\emptyset$.

If we think of a set as a box containing some stuff, then the empty set is a box with nothing in it.

**Definition 2.3.** The language associated to sets is specific.

$$S = \{x \in A : x \text{ satisfies some condition}\}$$

The first part "$x \in A$" denotes what type of $x$ is being considered. The statements to the right of the colon are the conditions that $x$ must satisfy in order to be members of the set. This notation is read as "The set of all $x$ in $A$ such that $x$ satisfies some condition," where "some condition" is something specific about the restrictions on $x$ relative to $A$. This notation is often called **set builder notation**.

**Exercise 2.4.** Unpack each of the following sets into a description using a sentence and see if you can determine exactly what elements each set contains.

1. $M = \{x \in \mathbb{R} : x \geq 2\}$
2. $A = \{x \in \mathbb{N} : x = 3k \text{ for some } k \in \mathbb{N}\}$
3. $T = \{t \in \mathbb{R} : t^2 \leq 2\}$
4. $H = \{t \in \mathbb{R} : t = 1 - \frac{1}{n}, \text{ where } n \in \mathbb{Z}\}$

**Exercise 2.5.** Write each of the following sentences using set builder notation.

1. Suppose $R$ is the set of all real numbers $x$ such that $x$ is less than $-\sqrt{2}$.
2. Suppose $A$ is the set of all real numbers $y$, such that $y$ is greater than $-12$ and less than $42.4$.
3. Suppose $D$ is the set of all even natural numbers.

**Definition 2.6.** If $A$ and $B$ are sets, then we say that $A$ is a **subset** of $B$, written $A \subseteq B$, provided that every element of $A$ is also an element of $B$.

**Remark 2.7.** Observe that $A \subseteq B$ is equivalent to "For all $x$ (in the universe of discourse), if $x \in A$, then $x \in B$." Since we know how to deal with "for all" statements and conditional propositions, we know how to go about proving $A \subseteq B$.

**Question 2.8.** Suppose that $A$ and $B$ are sets. Describe a general strategy for proving that $A \subseteq B$.

**Theorem 2.9.** Let $S$ be a set. Then

This work is an adaptation of notes written by Stan Yoshinobu of Cal Poly and Matthew Jones of California State University, Dominguez Hills.

1. $S \subseteq S$,
2. $\emptyset \subseteq S$.

**Exercise 2.10.** List all of the subsets of $A = \{1, 2, 3\}$. Any conjectures about how many there might be for a set with $n$ elements?

**Theorem 2.11** (*, Transitivity of subsets)**.** Suppose that $A$, $B$, and $C$ are sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

**Definition 2.12.** If $A \subseteq B$, then $A$ is called a **proper subset** provided that $A \neq B$. In this case, we may write $A \subset B$ or $A \subsetneq B$.*

**Definition 2.13** (Interval Notation)**.** For $a, b \in \mathbb{R}$ with $a < b$, we define the following.

1. $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
2. $(a, \infty) = \{x \in \mathbb{R} : a < x\}$
3. $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$
4. $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

We analogously define $[a, b)$, $(a, b]$, $[a, \infty)$, and $(-\infty, b]$.

**Exercise 2.14.** Provide two examples of proper subsets of the interval $[0, 1]$.

Here are some more definitions. In each case, take $U$ to be the universe of discourse.

**Definition 2.15.** The **union** of the sets $A$ and $B$ is $A \cup B = \{x \in U : x \in A \text{ or } x \in B\}$

**Definition 2.16.** The **intersection** of the sets $A$ and $B$ is $A \cap B = \{x \in U : x \in A \text{ and } x \in B\}$

**Definition 2.17.** The **set difference** of the sets $A$ and $B$ is $A \setminus B = \{x \in U : x \in A \text{ and } x \notin B\}$

**Definition 2.18.** The **complement of** $A$ (relative to $U$) is the set $A^c = U \setminus A = \{x \in U : x \notin A\}$

**Definition 2.19.** If two sets $A$ and $B$ have the property that $A \cap B = \emptyset$, then we say that $A$ and $B$ are **disjoint** sets.

**Exercise 2.20.** Suppose that the universe of discourse is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 5\}$, and $C = \{2, 4, 6, 8\}$. Find each of the following.

1. $A \cap C$
2. $A \cap B$
3. $A \cup C$
4. $A \cup B$
5. $A \setminus B$
6. $B \setminus A$
7. $C \setminus B$

* *Warning:* Some books use $\subset$ to mean $\subseteq$.

8. $B \cap C$

9. $B^c$

10. $A^c$

11. $(A \cup B)^c$

12. $A^c \cap B^c$

**Exercise 2.21.** Suppose that the universe of discourse is $U = \mathbb{R}$. Let $A = [-3, -1)$ and $B = (-2.5, 2)$, $C = (-2, 0]$. Find each of the following.

1. $A^c$

2. $A \cap C$

3. $A \cap B$

4. $A \cup C$

5. $A \cup B$

6. $(A \cap B)^c$

7. $(A \cup B)^c$

8. $A \setminus B$

9. $A \setminus (B \cup C)$

10. $B \setminus A$

11. $B \cap C$

**Theorem 2.22** (*)**.** Let $A$ and $B$ be sets. If $A \subseteq B$, then $B^c \subseteq A^c$.

**Definition 2.23.** Two sets $A$ and $B$ are **equal** if and only if $A \subseteq B$ and $B \subseteq A$. In this case we write $A = B$.

**Remark 2.24.** Given two sets $A$ and $B$, if we want to prove $A = B$, then we have to do two separate "mini" proofs: one for $A \subseteq B$ and one for $B \subseteq A$.

**Theorem 2.25** (*)**.** Let $A$ and $B$ be sets. Then $A \setminus B = A \cap B^c$.

**Theorem 2.26** (*, DeMorgan's Law)**.** Let $A$ and $B$ be sets. Then

1. $(A \cup B)^c = A^c \cap B^c$,

2. $(A \cap B)^c = A^c \cup B^c$.

(You only need to prove one of these; the other is similar.)

**Theorem 2.27** (*, Distribution of Union and Intersection)**.** Let $A$, $B$, and $C$ be sets. Then

1. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,

2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(You only need to prove one of these; the other is similar.)