

4 Relations and Functions

4.1 Relations

Definition 4.1. An **ordered pair** is an object of the form (x, y) . Two ordered pairs (x, y) and (a, b) are **equal** if $x = a$ and $y = b$.

Definition 4.2. An **n -tuple** is object of the form (x_1, x_2, \dots, x_n) . Each x_i is referred to as the i th **component**.

Note that an ordered pair is just a 2-tuple.

Definition 4.3. If X and Y are sets, the **Cartesian product** of X and Y is defined by

$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

That is, $X \times Y$ is the set of all ordered pairs where the first element is from X and the second element is from Y . The set $X \times X$ is sometimes denoted by X^2 . We similarly define the Cartesian product of n sets, say X_1, \dots, X_n , by

$$\prod_{i=1}^n X_i = X_1 \times \dots \times X_n = \{(x_1, \dots, x_n) : \text{each } x_i \in X_i\}.$$

Example 4.4. Let $A = \{a, b, c\}$ and $B = \{\odot, \ominus\}$. Then

$$A \times B = \{(a, \odot), (a, \ominus), (b, \odot), (b, \ominus), (c, \odot), (c, \ominus)\}.$$

Exercise 4.5. Using the sets A and B from the previous example, find $B \times A$.

Exercise 4.6. Using the set B from the previous examples, find $B \times B$.

Exercise 4.7. What general conclusion can you make about $X \times Y$ versus $Y \times X$? When will they be equal?

Exercise 4.8. If X and Y are both finite sets, then how many elements will $X \times Y$ have? Be as specific as possible.

Exercise 4.9. Let $A = \{1, 2, 3\}$, $B = \{1, 2\}$, and $C = \{1, 3\}$. List the elements of the set $A \times B \times C$.

Exercise 4.10. Let $A = \mathbb{N}$ and $B = \mathbb{R}$. Describe the elements of the set $A \times B$.

Exercise 4.11. Let A be the set of all differentiable functions on the open interval $(0, 1)$, and let B equal the set of all derivatives of functions in A evaluated at $x = \frac{1}{2}$. Describe the elements of the set $A \times B$.

Exercise 4.12. Three space, \mathbb{R}^3 , is a Cartesian product. Unpack the meaning of \mathbb{R}^3 using the Cartesian product, and write the complete set notation version.

Exercise 4.13. Let $X = [0, 1]$ and let $Y = \{1\}$. Describe geometrically what $X \times Y$, $Y \times X$, $X \times X$, and $Y \times Y$ look like.

Definition 4.14. Let X and Y be sets. A **relation** from a set X to a set Y is a subset of $X \times Y$. A relation on X is a subset of $X \times X$.

Example 4.15. You may not realize it, but you are familiar with many relations. For example, on the real numbers, we have the relation \leq . We could say that $(3, \pi)$ is in the relation since $3 \leq \pi$. However, $(1, -1)$ is not in the relation since $1 \not\leq -1$. (Order matters!)

Remark 4.16. Different notations for relations are used in different contexts. When talking about relations in the abstract, we indicate that a pair (a, b) is in the relation by some notation like $a \sim b$, which is read “ a is related to b .”

Example 4.17. Let P_f denote the set of all people with accounts on Facebook. Define F via xFy iff x is friends with y . Then F is a relation on P_f .

Remark 4.18. We can often represent relations using graphs or digraphs. Given a finite set X and a relation \sim on X , a **digraph** (short for *directed graph*) is a discrete graph having the members of X as vertices and a directed edge from x to y iff $x \sim y$.

Example 4.19. Let $A = \{a, b, c\}$ and define $\sim = \{(a, a), (a, b), (b, c), (c, b), (c, a)\}$. The digraph for \sim is a graph with vertices a, b, c and the following arrows: a to a , a to b , b to c , c to b , c to a .

Exercise 4.20. Let $A = \{1, 2, 3, 4, 5, 6\}$ Define $|$ on A via $x|y$ iff x divides y . Draw the digraph for $|$ on A .

When X or Y is infinite, it is not practical to draw a digraph. However, you are familiar with the graphs of some relations involving infinite sets.

Example 4.21. When we write $x^2 + y^2 = 1$, we are implicitly defining a relation. In particular, the relation is the set of ordered pairs (x, y) satisfying $x^2 + y^2 = 1$. In set notation:

$$\{(x, y) : x^2 + y^2 = 1\}$$

The graph of this relation in \mathbb{R}^2 is the standard unit circle.

Exercise 4.22. Define \sim on \mathbb{R} via $x \sim y$ iff $x \leq y$. Draw a picture of this relation in \mathbb{R}^2 .

Definition 4.23. Let \sim be a relation on a set A .

1. \sim is **reflexive** if for all $x \in A$, $x \sim x$ (every element is related to itself).
2. \sim is **symmetric** if for all $x, y \in A$, if $x \sim y$, then $y \sim x$.
3. \sim is **transitive** if for all $x, y, z \in A$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

Example 4.24.

1. \leq on \mathbb{R} is reflexive and transitive, but not symmetric. $<$ on \mathbb{R} is transitive, but not symmetric and not reflexive.
2. If S is a set, then \subseteq on $\mathcal{P}(S)$ is reflexive and transitive, but not symmetric.
3. $=$ on \mathbb{R} is reflexive, symmetric, and transitive.

Exercise 4.25. Given a finite set A and a relation \sim , describe what each of reflexive, symmetric, and transitive look like in terms of a digraph.

Exercise 4.26. Let P be the set of people at a party and define N via $(x, y) \in N$ iff x knows the name of y . Describe what it would mean for N to be reflexive, symmetric, and transitive.

Exercise 4.27. Determine whether each of the following relations is reflexive, symmetric, or transitive.

1. Let P_f denote the set of all people with accounts on Facebook. Define F via xFy iff x is friends with y .
2. Let P be the set of all people and define H via xHy iff x and y have the same height.
3. Let P be the set of all people and define T via xTy iff x is taller than y .
4. Consider the relation “divides” on \mathbb{N} .
5. Let L be the set of lines and define \parallel via $l_1 \parallel l_2$ iff l_1 is parallel to l_2 .
6. Let $C[0, 1]$ be the set of continuous functions on $[0, 1]$. Define $f \sim g$ iff

$$\int_0^1 |f(x)| dx = \int_0^1 |g(x)| dx.$$

7. Define \sim on \mathbb{N} via $n \sim m$ iff $n + m$ is even.
8. Define D on \mathbb{R} via $(x, y) \in D$ iff $x = 2y$.