4 Relations and Functions (continued)

4.2 Equivalence Relations

Remark 4.28. So that we have them handy, let's recall the following definitions. Let $\sim$ be a relation on a set A. Then

1. $\sim$ is **reflexive** if for all $x \in A$, $x \sim x$ (every element is related to itself).

2. $\sim$ is **symmetric** if for all $x, y \in A$, if $x \sim y$, then $y \sim x$.

3. $\sim$ is **transitive** if for all $x, y, z \in A$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

As we've seen in the previous section of notes, these conditions are mutually exclusive. That is, a relation may have some combination of these properties, but not necessarily all of them. However, we have a special name for when a relation does satisfy all three.

Definition 4.29. Let $\sim$ be a relation on a set A. Then $\sim$ is called an **equivalence relation** if $\sim$ is reflexive, symmetric, and transitive.

Exercise 4.30. Given a finite set A and a relation $\sim$ on A, describe what the corresponding digraph would have to look like in order for $\sim$ to be an equivalence relation.

Exercise 4.31. Let $A = \{a, b, c, d, e\}$. Make up an equivalence relation on A by drawing a digraph such that a is not related b and c is not related to b.

Exercise 4.32. Let $S = \{1, 2, 3, 4, 5, 6\}$ and define

$$\sim = \{(1,1),(1,6),(2,2),(2,3),(2,4),(3,3),(3,2),(3,4),(4,4),(4,2),(4,3),(5,5),(6,6),(6,1)\}.$$

Justify that this is an equivalence relation.

Problem 4.33. Determine which of the following are equivalence relations. Some of these occurred in the last section of notes and you are welcome to use your answers from those problems.

1. Let P_f denote the set of all people with accounts on Facebook. Define F via xFy iff x is friends with y.

2. Let P be the set of all people and define H via xHy iff x and y have the same height.

3. Let P be the set of all people and define T via xTy iff x is taller than y.

4. Consider the relation "divides" on $\mathbb{N}$.

5. Let L be the set of lines and define $||$ via $l_1||l_2$ iff l_1 is parallel to l_2.

6. Let $C[0, 1]$ be the set of continuous functions on $[0, 1]$. Define $f \sim g$ iff
$$\int_0^1 |f(x)|\, dx = \int_0^1 |g(x)|\, dx.$$

7. Define $\sim$ on $\mathbb{N}$ via $n \sim m$ iff $n + m$ is even.

8. Define D on $\mathbb{R}$ via $(x, y) \in D$ iff $x = 2y$.

9. Define $\sim$ on $\mathbb{Z}$ via $a \sim b$ iff $a - b$ is a multiple of 5.
10. Define $\sim$ on $\mathbb{R}^2$ via $(x_1, y_1) \sim (x_2, y_2)$ iff $x_1^2 + y_1^2 = x_2^2 + y_2^2$.
11. Define $\sim$ on $\mathbb{R}$ via $x \sim y$ iff $\lfloor x \rfloor = \lfloor y \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x (e.g., $\lfloor \pi \rfloor = 3$, $\lfloor -1.5 \rfloor = -2$, and $\lfloor 4 \rfloor = 4$).
12. Define $\sim$ on $\mathbb{R}$ via $x \sim y$ iff $|x - y| < 1$.

Definition 4.34. Let $\sim$ be a relation on a set A (not necessarily an equivalence relation) and let $x \in A$. Then we define the **set of relatives of** x via

$$R_x = \{y \in A : x \sim y\}.$$

Also, define

$$\Omega_\sim = \{R_x : x \in A\}.$$

Notice that $\Omega_\sim$ is a set of sets. In particular, an element in $\Omega_\sim$ is a subset of A.

Exercise 4.35. Let P_f and F be as in part 1 of Exercise 4.33. Describe R_{Bob} (assume you know which Bob we're talking about). What is Ω_F?

Exercise 4.36. Using your digraph in Exercise 4.31, find $\Omega_\sim$ for all $x \in A$.

Exercise 4.37. Consider the relation $\leq$ on $\mathbb{R}$. If $x \in \mathbb{R}$, what is R_x?

Exercise 4.38. Find R_1 and R_2 for the relation given in part 9 of Exercise 4.33. How many different sets of relatives are there? What are they?

Exercise 4.39. Find R_x for all $x \in S$ for S and $\sim$ from Exercise 4.32. Any observations?

Theorem 4.40 (*)**.** Suppose $\sim$ is an equivalence relation on a set A and let $a, b \in A$. Then $R_a = R_b$ iff $a \sim b$.

Theorem 4.41 (*)**.** Suppose $\sim$ is an equivalence relation on a set A. Then

1. $\bigcup_{x \in A} R_x = A$, and
2. for all $x, y \in A$, either $R_x = R_y$ or $R_x \cap R_y = \emptyset$.

Definition 4.42. In light of Theorem 4.41, if $\sim$ is an equivalence relation on a set A, then we refer to each R_x as the **equivalence class** of x. In this case, $\Omega_\sim$ is the set of equivalence classes determined by $\sim$.

Remark 4.43. The upshot of Theorem 4.41 is that given an equivalence relation, every element lives in exactly one equivalence class. We'll see in the next section of notes that we can run this in reverse. That is, if we separate out the elements of a set so that every element is an element of exactly one subset (like the bins of my kid's toys), then this determines an equivalence relation. More on this later.

Example 4.44. The set of relatives that you found in part 9 of Exercise 4.33 is the set of equivalence classes modulo 5.

Exercise 4.45. If $\sim$ is an equivalence relation on a finite set A, then what is the connection between the equivalence classes and the corresponding digraph?

Exercise 4.46. For each of the equivalence relations in Exercise 4.33, describe the equivalence classes as best as you can.