Braid graphs in simply-laced triangle-free Coxeter systems are median

CU Lie Theory Seminar

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Coxeter systems

Definition

A Coxeter system consists of a group W (called a Coxeter group) generated by a set S of involutions with presentation

$$W = \langle S \mid s^2 = e, \quad (st)^{m(s,t)} = e \rangle,$$

where $m(s,t) \geq 2$ for $s \neq t$.

Comments

- The elements of S are distinct as group elements.
- $m(s,t)$ is the order of st.

Coxeter systems (continued)

Since s and t are involutions, the relation $(st)^{m(s,t)} = e$ can be rewritten:

$$\left.\begin{array}{lcr} m(s,t) = 2 & \implies & st = ts \end{array}\right\} \quad \text{commutation relation}$$

$$\left.\begin{array}{lcr} m(s,t) = 3 & \implies & sts = tst \\ m(s,t) = 4 & \implies & stst = tsts \\ & \vdots & \end{array}\right\} \quad \text{braid relations}$$

This allows the replacement

$$\underbrace{sts\cdots}_{m(s,t)} \mapsto \underbrace{tst\cdots}_{m(s,t)}$$

in any word, which is called a commutation move if $m(s,t) = 2$ and a braid move if $m(s,t) \geq 3$.

Coxeter graphs

Definition

We can encode (W, S) with a unique Coxeter graph Γ having:

- Vertex set $= S$
- $\{s, t\}$ edge labeled with $m(s, t)$ whenever $m(s, t) \geq 3$

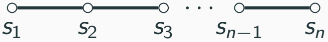
Comments

- Typically labels of $m(s, t) = 3$ are omitted.
- Edges correspond to non-commuting pairs of generators.
- If all $m(s, t) \leq 3$, then Γ and W are called simply laced.
- If Γ has no 3-cycles, then Γ and W are called triangle free.
- If both simply laced and triangle free, then Γ and W are of type Λ.

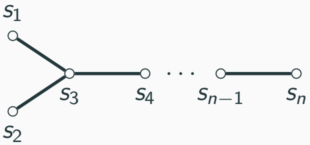
Coxeter graphs (continued)

Example

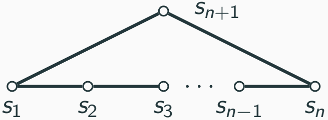
Here are Coxeter graphs for four common simply-laced Coxeter systems. With the exception of $\widetilde{A}_2$ (3-cycle), the rest are of type Λ.



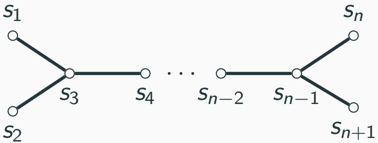
A_n



D_n



$\widetilde{A}_n$



$\widetilde{D}_n$

The top two Coxeter graphs yield finite groups while the bottom two yield infinite groups.

Reduced expressions & Matsumoto's Theorem

Definition

A word $\boldsymbol{\alpha} = s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ is called an expression for w if it is equal to w when considered as a group element. If m is minimal among all expressions for w, $\boldsymbol{\alpha}$ is a called a reduced expression, and w has length $\ell(w) := m$.

$$\mathcal{R}(w) = \text{set of reduced expressions for } w$$

A factor of $\boldsymbol{\alpha}$ is a word of the form $\boldsymbol{\beta} = s_{x_i} s_{x_{i+1}} \cdots s_{x_{j-1}} s_{x_j}$ for $1 \leq i \leq j \leq m$. We write $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$.

Matsumoto's Theorem

Any two reduced expressions for $w \in W$ differ by a sequence of commutation & braid moves.

Matsumoto graphs

Definition

For $w \in W$, define the Matsumoto graph $\mathcal{M}(w)$ via:

- Vertex set $= \mathcal{R}(w)$
- $\{\alpha, \beta\}$ iff α and β are related via a commutation or braid move

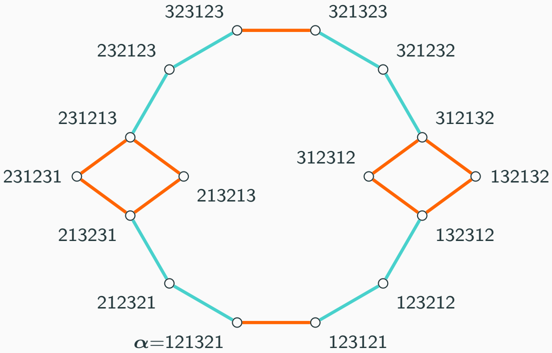
Comments

- Matsumoto's Theorem implies that $\mathcal{M}(w)$ is connected.
- Every cycle in a Matsumoto graph has even length (Bergeron, Ceballos, Labbé / Grinberg, Postnikov).
- Every Matsumoto graph is bipartite.

Matsumoto graphs (continued)

Example

Consider reduced expression $\boldsymbol{\alpha} = 121321$ for $w \in W(A_3)$. Then $\mathcal{M}(w)$ is as follows:



Braid equivalence & Braid graphs

Definition

If $\alpha, \beta \in \mathcal{R}(w)$, then α and β are braid equivalent iff α and β are related by a sequence of braid moves. We write $\alpha \sim \beta$.

Comments

- Braid equivalence is an equivalence relation.
- Equivalence classes are called braid classes, denoted $[\alpha]$.

Definition

We can encode a braid class $[\alpha]$ in a braid graph, denoted $\mathcal{B}(\alpha)$:

- Vertex set $= [\alpha]$
- $\{\gamma, \beta\}$ iff γ and β are related via a single braid move

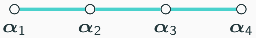
Braid graphs are the maximal blue connected components in the Matsumoto graph.

Braid graphs (continued)

Example

Consider Coxeter system of type A_4. The braid class for the reduced expression $\boldsymbol{\alpha}_1 = 1213243$ consists of the following reduced expressions:

$$\boldsymbol{\alpha}_1 = \underline{121}3243,\ \boldsymbol{\alpha}_2 = \underline{21\overline{2}}\overline{32}43,\ \boldsymbol{\alpha}_3 = 21\overline{32\underline{3}}\underline{43},\ \boldsymbol{\alpha}_4 = 2132\underline{434}.$$



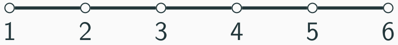
Braid graphs (continued)

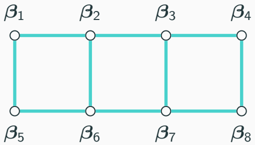
Example

In the Coxeter system of type A_6, the expression $\boldsymbol{\beta}_1 = 1213243565$ is reduced. Its braid class consists of the following reduced expressions:

$\boldsymbol{\beta}_1 = \underline{121}3243\underline{565}$, $\boldsymbol{\beta}_2 = \underline{21\overline{2}}\overline{32}43\underline{565}$, $\boldsymbol{\beta}_3 = 21\underline{32\overline{3}}\overline{43}\underline{565}$, $\boldsymbol{\beta}_4 = 2132\overline{434}\underline{565}$,

$\boldsymbol{\beta}_5 = \underline{121}3243\underline{656}$, $\boldsymbol{\beta}_6 = \underline{21\overline{2}}\overline{32}43\underline{656}$, $\boldsymbol{\beta}_7 = 21\underline{32\overline{3}}\overline{43}\underline{656}$, $\boldsymbol{\beta}_8 = 2132\overline{434}\underline{656}$.



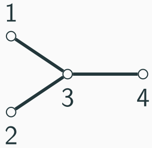


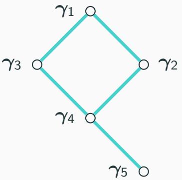
Braid graphs (continued)

Example

Consider Coxeter system of type D_4. The expression $\gamma_1 = 2321434$ is reduced and its braid class consists of the following reduced expressions:

$$\gamma_1 = \underline{434}1\underline{232},\ \gamma_2 = \underline{343}1\underline{232},\ \gamma_3 = \underline{434}1\underline{323},\ \gamma_4 = \underline{34\overline{3}}\,\overline{1}\,\underline{\overline{3}23},\ \gamma_5 = 34\underline{131}23.$$





Local support of reduced expressions

Notation

For $i \leq j$, we define the interval

$$[\![i, j]\!] := \{i, i+1, \ldots, j-1, j\}.$$

Definition

If $\boldsymbol{\alpha} = s_{x_1} s_{x_2} \cdots s_{x_m}$ is a reduced expression, we define:

- $\boldsymbol{\alpha}_{[\![i,j]\!]} := s_{x_i} s_{x_{i+1}} \cdots s_{x_{j-1}} s_{x_j}$ (factor of $\boldsymbol{\alpha}$).
- Local support of $\boldsymbol{\alpha}$ over $[\![i, j]\!]$:

$$\mathrm{supp}_{[\![i,j]\!]}(\boldsymbol{\alpha}) := \{s_{x_k} \mid k \in [\![i, j]\!]\}.$$

- Local support of the braid class $[\boldsymbol{\alpha}]$ over $[\![i, j]\!]$:

$$\mathrm{supp}_{[\![i,j]\!]}([\boldsymbol{\alpha}]) := \bigcup_{\boldsymbol{\beta} \in [\boldsymbol{\alpha}]} \mathrm{supp}_{[\![i,j]\!]}(\boldsymbol{\beta}).$$

Braid shadows

Important!

We assume all Coxeter systems are simply laced, often of type Λ.

Definition

Let $\boldsymbol{\alpha}$ be a reduced expression.

- $[\![i, i+2]\!]$ is braid shadow for $\boldsymbol{\alpha}$ if $\alpha_{[\![i,i+2]\!]} = sts$ with $m(s,t) = 3$.
- Set of braid shadows for $\boldsymbol{\alpha}$ denoted by $\mathcal{S}(\boldsymbol{\alpha})$.
- Collection of braid shadows for braid class $[\boldsymbol{\alpha}]$ is given by

$$\mathcal{S}([\boldsymbol{\alpha}]) := \bigcup_{\boldsymbol{\beta} \in [\boldsymbol{\alpha}]} \mathcal{S}(\boldsymbol{\beta}).$$

- If $[\![i, i+2]\!]$ is a braid shadow for $[\boldsymbol{\alpha}]$, then position $i+1$ (in any reduced expression in $[\boldsymbol{\alpha}]$) is called the center of the braid shadow.
- Cardinality of $\mathcal{S}([\boldsymbol{\alpha}])$ is rank of $\boldsymbol{\alpha}$, denoted by $\operatorname{rank}(\boldsymbol{\alpha})$.

Links and braid chains

Theorem

If $\boldsymbol{\alpha}$ is a reduced expression, then

$$[\![i, i+2]\!] \in \mathcal{S}([\boldsymbol{\alpha}]) \implies [\![i+1, i+3]\!] \notin \mathcal{S}([\boldsymbol{\alpha}]).$$

Upshot: braid shadows are either disjoint or overlap by a single position.

Definition

Let $\boldsymbol{\alpha} = s_{x_1} s_{x_2} \cdots s_{x_m}$ be a reduced expression.

- $\boldsymbol{\alpha}$ is a link provided either $m = 1$ or m is odd and

$$\mathcal{S}([\boldsymbol{\alpha}]) = \{[\![1,3]\!], [\![3,5]\!], \ldots, [\![m-4, m-2]\!], [\![m-2, m]\!]\}.$$

- If $\boldsymbol{\alpha}$ is a link, then corresponding braid class is called a braid chain.

Loosely speaking, $\boldsymbol{\alpha}$ is link if there is a sequence of overlapping braid moves that "cover" the positions $1, 2, \ldots, m$.

Links and braid chains (continued)

Example

Recall the reduced expression $\boldsymbol{\alpha}_1 = 1213243$ in the Coxeter system of type A_4 with braid class:

$$\boldsymbol{\alpha}_1 = \underline{121}3243,\ \boldsymbol{\alpha}_2 = \underline{21\overline{2}}\overline{32}43,\ \boldsymbol{\alpha}_3 = 21\overline{32}\underline{\overline{3}43},\ \boldsymbol{\alpha}_4 = 2132\underline{434}.$$

By inspection, we see that

$$\mathcal{S}(\boldsymbol{\alpha}_1) = \{[\![1,3]\!]\} \quad \text{and} \quad \mathcal{S}([\boldsymbol{\alpha}_1]) = \{[\![1,3]\!], [\![3,5]\!], [\![5,7]\!]\}.$$

Hence $\boldsymbol{\alpha}_1$ is a link of rank 3 and $[\boldsymbol{\alpha}_1]$ is a braid chain

Links and braid chains (continued)

Example

Recall the reduced expression $\beta_1 = 1213243565$ in the Coxeter system of type A_6 with braid class:

$$\beta_1 = \underline{121}3243\underline{565},\ \beta_2 = \underline{21}\underline{\overline{2}}\overline{32}43\underline{565},\ \beta_3 = 21\underline{32}\underline{\overline{3}}\overline{43}\underline{565},\ \beta_4 = 2132\overline{434}\underline{565},$$

$$\beta_5 = \underline{121}3243\underline{656},\ \beta_6 = \underline{21}\underline{\overline{2}}\overline{32}43\underline{656},\ \beta_7 = 21\underline{32}\underline{\overline{3}}\overline{43}\underline{656},\ \beta_8 = 2132\overline{434}\underline{656}.$$

We see that

$$\mathcal{S}(\beta_1) = \{[\![1,3]\!], [\![8,10]\!]\} \text{ and } \mathcal{S}([\beta_1]) = \{[\![1,3]\!], [\![3,5]\!], [\![5,7]\!], [\![8,10]\!]\},$$

It follows that β_1 is not a link. However, it turns out that the factors 1213243 and 565 of β_1 are links in their own right.

Example

Recall the reduced expression $\gamma_1 = 2321434$ in the Coxeter system of type D_4 with braid class:

$$\gamma_1 = \underline{434}1\underline{232},\ \gamma_2 = \underline{343}1\underline{232},\ \gamma_3 = \underline{434}1\underline{323},\ \gamma_4 = \underline{34\overline{3}}\overline{1}\underline{\overline{3}23},\ \gamma_5 = 34\underline{131}23.$$

We see that

$$\mathcal{S}(\gamma_1) = \{[\![1,3]\!],[\![5,7]\!]\} \text{ and } \mathcal{S}([\gamma_1]) = \{[\![1,3]\!],[\![3,5]\!],[\![5,7]\!]\}.$$

So, γ_1 is a link of rank 3 and $[\gamma_1]$ is a braid chain. The link γ_4 is an example of something special called a Fibonacci link (braid graph is a Fibonacci cube).

Link factorization for reduced expressions

Definition

If $\boldsymbol{\alpha}$ is a reduced expression for $w \in W$ with $\ell(w) \geq 1$, then $\boldsymbol{\beta}$ is a link factor of $\boldsymbol{\alpha}$ provided:

- $\boldsymbol{\beta} \leq \boldsymbol{\alpha}$,
- $\boldsymbol{\beta}$ is a link, and
- If $\boldsymbol{\beta} < \boldsymbol{\gamma} \leq \boldsymbol{\alpha}$, then $\boldsymbol{\gamma}$ is not a link.

Theorem

Every reduced expression $\boldsymbol{\alpha}$ for a nonidentity group element can be written uniquely as a product of link factors, say $\boldsymbol{\alpha}_1 \boldsymbol{\alpha}_2 \cdots \boldsymbol{\alpha}_k$, where each $\boldsymbol{\alpha}_i$ is a link factor of $\boldsymbol{\alpha}$.

We refer to this product as the link factorization of $\boldsymbol{\alpha}$. For emphasis:

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 \mid \boldsymbol{\alpha}_2 \mid \cdots \mid \boldsymbol{\alpha}_k.$$

Link factorization across braid classes

Theorem

If $\boldsymbol{\alpha}$ is a reduced expression with link factorization $\boldsymbol{\alpha}_1 \mid \boldsymbol{\alpha}_2 \mid \cdots \mid \boldsymbol{\alpha}_k$, then

$$[\boldsymbol{\alpha}] = \left\{\boldsymbol{\beta}_1 \mid \boldsymbol{\beta}_2 \mid \cdots \mid \boldsymbol{\beta}_k : \boldsymbol{\beta}_i \in [\boldsymbol{\alpha}_i] \text{ for } 1 \leq i \leq k\right\}.$$

Moreover, the cardinality of the braid class for $\boldsymbol{\alpha}$ is given by

$$\operatorname{card}([\boldsymbol{\alpha}]) = \prod_{i=1}^{k} \operatorname{card}([\boldsymbol{\alpha}_i]),$$

and the rank of $\boldsymbol{\alpha}$ is given by

$$\operatorname{rank}(\boldsymbol{\alpha}) = \sum_{i=1}^{k} \operatorname{rank}(\boldsymbol{\alpha}_i).$$

Braid graphs for link factorizations

Corollary

If α is reduced expression with link factorization

$$\boldsymbol{\alpha} = \boldsymbol{\beta}_1 \mid \boldsymbol{\beta}_1 \mid \cdots \mid \boldsymbol{\beta}_m,$$

then $\mathcal{B}(\boldsymbol{\alpha})$ is the box product of the braid graphs for each $\boldsymbol{\beta}_i$.

Comment

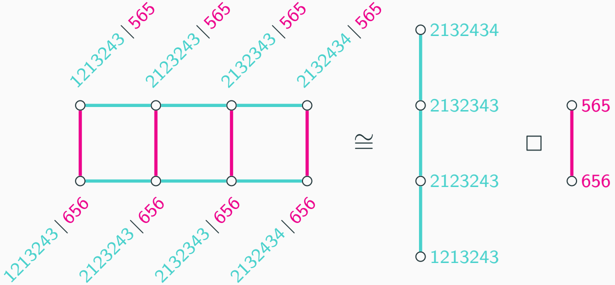
- Upshot: if you want to understand the structure of braid graphs, you can first characterize braid graphs for links.
- In the case of type A_n, links have odd length and the corresponding braid graphs are paths.

Braid graphs for link factorizations

Example

Consider reduced expression $\beta_1 = 1213243565$ in type A_6 from earlier. It has link factorization:

$$1213243 \mid 565.$$

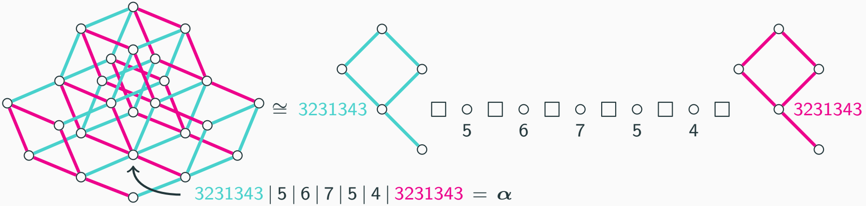


Braid graphs for link factorizations

Example

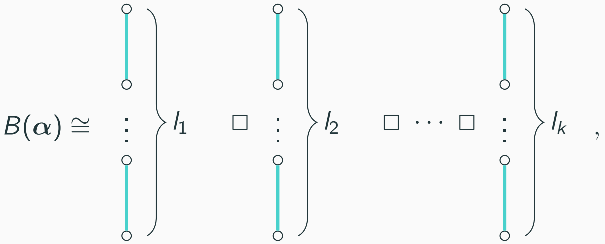
Consider reduced expression $\boldsymbol{\alpha} = 3231343567543231343$ in type D_7. It has link factorization:

$$3231343 \mid 5 \mid 6 \mid 7 \mid 5 \mid 4 \mid 3231343.$$



Theorem

If α is reduced expression for nonidentity element in type A_n with link factorization $\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_k$ such that each α_i has $2l_i - 1$ letters, then



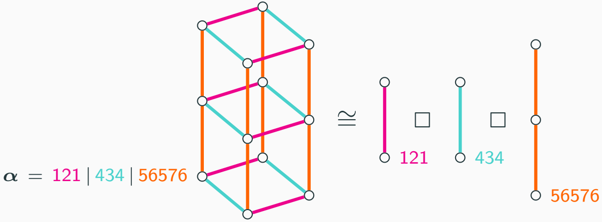
where ith link factor in the decomposition is a path graph with l_i vertices.

Braid graphs for link factorizations in type A_n (continued)

Example

Consider reduced expression $\alpha = 12143456576$ in type A_7 with link factorization:

$$121 \mid 434 \mid 56576.$$

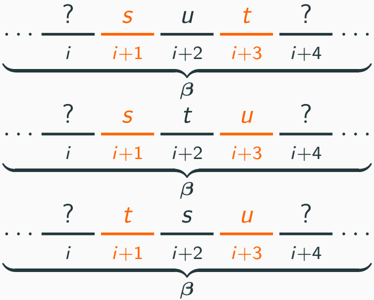


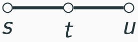
Facts about braid shadows

Theorem

Suppose (W, S) is of type Λ and let $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ be links of rank at least one.

- If $[\![i, i+2]\!] \in \mathcal{S}(\boldsymbol{\alpha}) \cap \mathcal{S}(\boldsymbol{\beta})$, then $\mathrm{supp}_{[\![i,i+2]\!]}(\boldsymbol{\alpha}) = \mathrm{supp}_{[\![i,i+2]\!]}(\boldsymbol{\beta})$.
- If $[\![i, i+2]\!] \in \mathcal{S}(\boldsymbol{\alpha})$, then $\mathrm{supp}_{[\![i,i+2]\!]}(\boldsymbol{\alpha}) = \{s, t\}$ with $m(s,t) = 3$ and $\mathrm{supp}_{[\![i+1]\!]}([\boldsymbol{\alpha}]) = \{s, t\}$.
- If additionally $[\![i+2, i+4]\!] \in \mathcal{S}(\boldsymbol{\alpha})$, then $\mathrm{supp}_{[\![i+2,i+4]\!]}(\boldsymbol{\alpha}) = \{t, u\}$ and $\mathrm{supp}_{[\![i+3]\!]}([\boldsymbol{\alpha}]) = \{t, u\}$ with $m(t,u) = 3$, $m(s,u) = 2$.





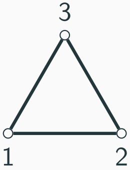
Why triangle free?

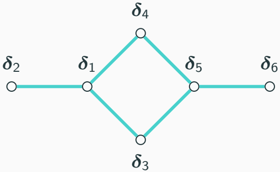
Example

Consider reduced expression $\boldsymbol{\delta}_1 = 1213121$ in type $\widetilde{A}_2$ with braid class:

$$\boldsymbol{\delta}_1 = \underline{12\overline{1}}\overline{\textcolor{orange}{3}}\underline{\overline{1}21},\ \boldsymbol{\delta}_2 = 12\overline{3\textcolor{orange}{1}3}21,\ \boldsymbol{\delta}_3 = \underline{212}\textcolor{orange}{3}\underline{121}$$

$$\boldsymbol{\delta}_4 = \underline{12\overline{1}}\overline{\textcolor{orange}{3}}\underline{\overline{2}12},\ \boldsymbol{\delta}_5 = \underline{21\overline{2}}\overline{\textcolor{orange}{3}}\underline{\overline{2}12},\ \boldsymbol{\delta}_6 = 21\overline{3\textcolor{orange}{2}3}12$$





Notice:

- $\mathrm{supp}_{[\![3,5]\!]}(\boldsymbol{\delta}_1) = \{1,3\} \neq \{2,3\} = \mathrm{supp}_{[\![3,5]\!]}(\boldsymbol{\delta}_5)$
- Cardinality of center of middle braid shadow is larger than 2.

Links are uniquely determined by signature

Definition

If (W, S) is of type Λ and $\boldsymbol{\alpha}$ is a link of rank r, the signature of $\boldsymbol{\alpha}$, denoted $\text{sig}(\boldsymbol{\alpha})$, is the ordered list of generators appearing in the centers of the braid shadows of $\boldsymbol{\alpha}$. Let $\text{sig}_i(\boldsymbol{\alpha})$ represent ith position of $\text{sig}(\boldsymbol{\alpha})$.

Theorem

Suppose (W, S) is of type Λ and let $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ be links. Then $\boldsymbol{\alpha} = \boldsymbol{\beta}$ iff $\text{sig}(\boldsymbol{\alpha}) = \text{sig}(\boldsymbol{\beta})$.

Upshot: Every link is uniquely determined by the generators appearing at the centers of the braid shadows.

Intervals in braid graphs

Definition

The interval between vertices u and v in a graph G, denoted $I(u, v)$, is the collection of vertices on any geodesic between u and v.

Definition

We define

$$\overline{\mathrm{sig}}(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \{\boldsymbol{x} \in [\boldsymbol{\alpha}] \mid \mathrm{sig}_i(\boldsymbol{x}) = \mathrm{sig}_i(\boldsymbol{\alpha}) \text{ if } \mathrm{sig}_i(\boldsymbol{\alpha}) = \mathrm{sig}_i(\boldsymbol{\beta})\}.$$

That is, $\overline{\mathrm{sig}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is the set of reduced expressions whose signatures agrees with common signatures of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

Theorem

If (W, S) is type Λ and $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ are links, then $I(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \overline{\mathrm{sig}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

Median graphs

Definition

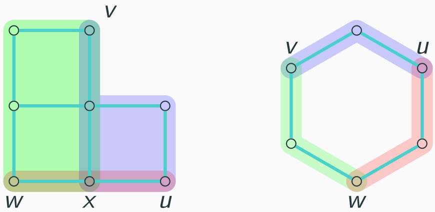
A connected graph is median if for any three vertices:

$$|\operatorname{med}(u, v, w) := I(u, v) \cap I(u, w) \cap I(v, w)| = 1.$$

That is, there is a unique vertex, called the median, that simultaneously lies on a geodesic between u and v, a geodesic between u and w, and a geodesic between v and w.

Example

The graph on the left is median while the one on the right is not.

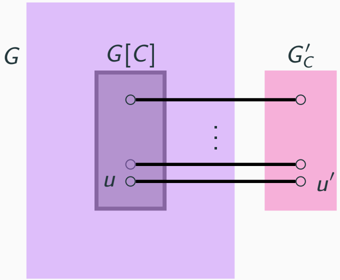


Median graphs (continued)

Definition

Given a graph G and a convex set $C \subseteq V(G)$, we define the expanded graph relative to C:

- Start with a graph G;
- Make an isomorphic copy of $G[C]$, denoted G'_C, where each $u \in C$ corresponds to $u' \in C' := V(G'_C)$;
- For each $u \in C$, join u and u' with an edge.



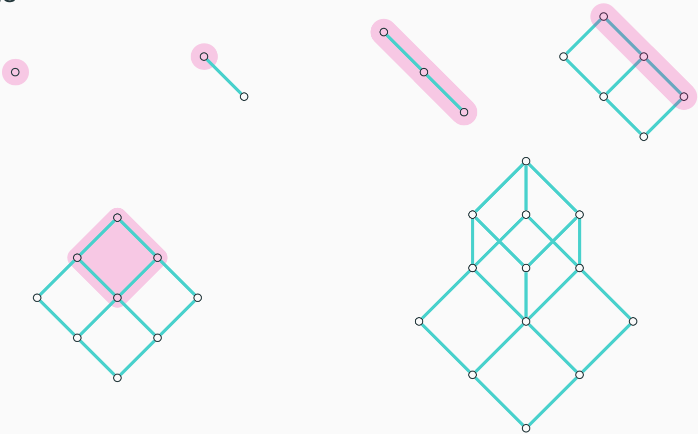
Median graphs (continued)

Median graphs (continued)

Proposition

A graph is median iff it can be obtained from a single vertex by a sequence of convex expansions.

Example



Tools for our main result

Notation

Given a reduced expression $\boldsymbol{\alpha}$, let $\hat{\boldsymbol{\alpha}}$ to be the expression obtained by deleting the rightmost two letters of $\boldsymbol{\alpha}$.

Warning!

Certainly, $\hat{\boldsymbol{\alpha}}$ is reduced but may not be a link!

Definition

Suppose $\boldsymbol{\alpha}$ is a link of rank $r \geq 1$ and let $\boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$:

$$X_{\boldsymbol{\sigma}} := \{\boldsymbol{\beta} \in [\boldsymbol{\alpha}] \mid \mathrm{sig}_r(\boldsymbol{\beta}) = \mathrm{sig}_r(\boldsymbol{\sigma})\}$$

$$Y_{\boldsymbol{\sigma}} := \{\boldsymbol{\beta} \in [\boldsymbol{\alpha}] \mid \mathrm{sig}_r(\boldsymbol{\beta}) \neq \mathrm{sig}_r(\boldsymbol{\sigma})\}$$

Theorem

If (W, S) is type Λ and $\boldsymbol{\alpha}$ is a link of rank $r \geq 2$, then there exists $\boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$ such that $[\![2r-3, 2r-1]\!], [\![2r-1, 2r+1]\!] \in \mathcal{S}(\boldsymbol{\sigma})$. In this case, $\hat{\boldsymbol{\sigma}}$ is a link of rank $r-1$.

Tools for our main result (continued)

Theorem

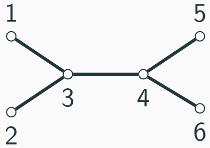
Suppose (W, S) is type Λ and $\boldsymbol{\alpha}$ is a link of rank $r \geq 2$. Choose $\boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$ according to previous theorem. Then

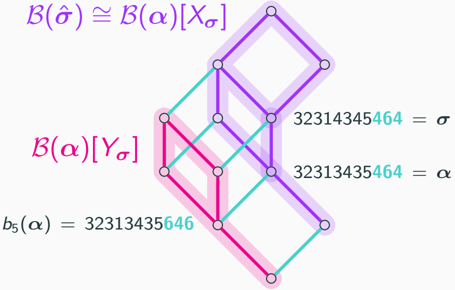
- $\{X_{\boldsymbol{\sigma}}, Y_{\boldsymbol{\sigma}}\}$ is a partition of $[\boldsymbol{\alpha}]$.
- $X_{\boldsymbol{\sigma}}$ and $Y_{\boldsymbol{\sigma}}$ are convex.
- $\boldsymbol{\beta} \in X_{\boldsymbol{\sigma}}$ iff $\hat{\boldsymbol{\beta}} \in [\hat{\boldsymbol{\sigma}}]$.
- If $\boldsymbol{\beta} \in Y_{\boldsymbol{\sigma}}$, then $[\![2r-1, 2r+1]\!] \in \mathcal{S}(\boldsymbol{\beta})$ and $\widehat{b_{2r}(\boldsymbol{\beta})} \in [\hat{\boldsymbol{\sigma}}]$.
- There exists an isometric embedding from $\mathcal{B}(\hat{\boldsymbol{\sigma}})$ into $\mathcal{B}(\boldsymbol{\alpha})$ whose image is $\mathcal{B}(\boldsymbol{\alpha})[X_{\boldsymbol{\sigma}}]$.
- $\mathcal{B}(\boldsymbol{\alpha})[Y_{\boldsymbol{\sigma}}]$ is an isometric subgraph of $B(\boldsymbol{\alpha})$.
- If $\boldsymbol{\beta} \in X_{\boldsymbol{\sigma}}$ and $\boldsymbol{\gamma} \in Y_{\boldsymbol{\sigma}}$, then $d(\boldsymbol{\beta}, \boldsymbol{\gamma}) = d(\boldsymbol{\beta}, b_{2r}(\boldsymbol{\gamma})) + 1$.

Visualizing previous result

Example

Consider link $\alpha = 32313435464$ in the Coxeter system of type $\widetilde{D}_5$.





Braid graphs for links are median

Theorem

If (W, S) is of type Λ and $\boldsymbol{\alpha}$ is a link, then $\mathcal{B}(\boldsymbol{\alpha})$ is median.

Outline of Proof

- We induct on rank. Base cases $r = 0$ and $r = 1$ check out.
- Suppose all braid graphs for links of rank $r - 1$ are median and consider a link $\boldsymbol{\alpha}$ or rank r.
- Choose $\boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$ with $[\![2r-3, 2r-1]\!], [\![2r-1, 2r+1]\!] \in \mathcal{S}(\boldsymbol{\sigma})$ according to earlier result.
- By induction $\mathcal{B}(\hat{\boldsymbol{\sigma}}) \cong \mathcal{B}(\boldsymbol{\alpha})[X_{\boldsymbol{\sigma}}]$ is median.
- The set $C := \{\boldsymbol{\beta} \in X_{\boldsymbol{\sigma}} \mid \mathrm{sig}_r(\boldsymbol{\beta})\,\mathrm{sig}_r(\boldsymbol{\sigma})\}$ is convex and $\mathcal{B}(\boldsymbol{\alpha})[C] \cong \mathcal{B}(\boldsymbol{\alpha})[Y_{\boldsymbol{\sigma}}]$ via $\boldsymbol{\mu} \mapsto b_r(\boldsymbol{\mu})$.
- It follows that $\mathcal{B}(\boldsymbol{\alpha})$ is obtained from $\mathcal{B}(\boldsymbol{\alpha})[X_{\boldsymbol{\sigma}}]$ via a convex expansion relative to C.

Signature majority determines median

Definition

We define the *i*th majority of links $\boldsymbol{\alpha} \sim \boldsymbol{\beta} \sim \boldsymbol{\sigma}$ of rank r via

$$\mathrm{maj}_i(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma}) := \begin{cases} \mathrm{sig}_i(\boldsymbol{\alpha}), & \text{if } \mathrm{sig}_i(\boldsymbol{\alpha}) = \mathrm{sig}_i(\boldsymbol{\beta}) \text{ or } \mathrm{sig}_i(\boldsymbol{\alpha}) = \mathrm{sig}_i(\boldsymbol{\sigma}) \\ \mathrm{sig}_i(\boldsymbol{\beta}), & \text{otherwise,} \end{cases}$$

and their majority via

$$\mathrm{maj}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma}) := (\mathrm{maj}_1(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma}), \ldots, \mathrm{maj}_r(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma})).$$

Corollary

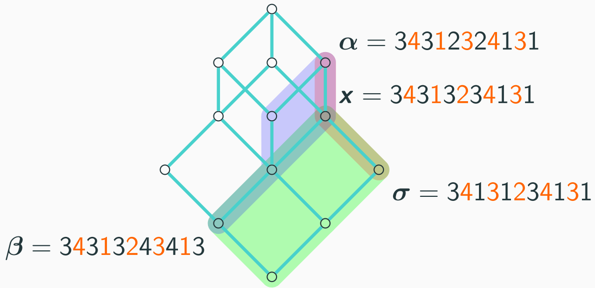
If (W, S) is type Λ, then the median of links $\boldsymbol{\alpha} \sim \boldsymbol{\beta} \sim \boldsymbol{\sigma}$ is the unique link $\boldsymbol{x}$ satisfying

$$\mathrm{sig}(\boldsymbol{x}) = \mathrm{maj}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma}).$$

Signature majority determines median (continued)

Example

Consider braid equivalent links $\boldsymbol{\alpha} = 34312324131$, $\boldsymbol{\beta} = 34313243413$, and $\boldsymbol{\sigma} = 34131234131$ in $[\boldsymbol{\alpha}]$ in Coxeter system of type D_4.



We see that

$$\text{maj}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\sigma}) = (4, 1, 2, 4, 3),$$

which corresponds to the signature of $\boldsymbol{x} = 34313234131$ in $[\boldsymbol{\alpha}]$.

Braid graphs for reduced expressions are median

Proposition

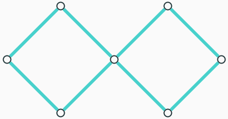
If graphs G_1 and G_2 are median, then $G_1 \square G_2$ is also median.

Theorem

If (W, S) is type Λ and α is any reduced expression, $\mathcal{B}(\alpha)$ is median.

Example

Not every median graph can be realized as the braid graph for a reduced expression! This graph is median but does not correspond to a braid graph in a type Λ Coxeter system.



Upshot: Braid graphs are "special" median graphs. What is "special"???

Partial cubes

If $n \in \mathbb{N} \cup \{0\}$, then we define the set of binary strings of length n as:

$$\{0,1\}^n := \{a_1 a_2 \cdots a_n \mid a_k \in \{0,1\}\}.$$

Definition

The hypercube of dimension n, denoted Q_n, is the graph with vertex set $V(Q_n) = \{0,1\}^n$ and two vertices are adjacent when their corresponding binary strings differ by exactly one digit.

Definition

A graph G is a partial cube if it can be isometrically embedded in some hypercube Q_n. The isometric dimension $\dim_I(G)$ of a partial cube is the minimum dimension of the hypercube into which the partial cube can be isometrically embedded.

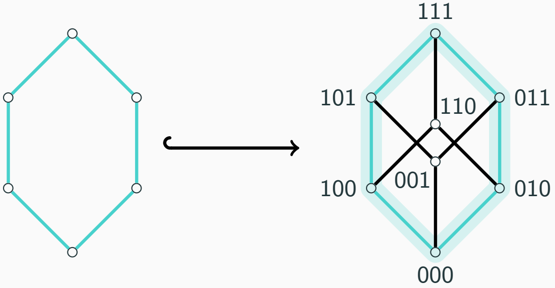
Partial cubes (continued)

Proposition

- If G_1 and G_2 are partial cubes, then $G_1 \square G_2$ is a partial cube with $\dim_I(G_1 \square G_2) = \dim_I(G_1) + \dim_I(G_2)$.
- Every median graph is a partial cube.

Example

The converse of second bullet is not true! We saw earlier that C_6 is not median. But it is a partial cube with isometric dimension 3.



Braid graphs are partial cubes

Theorem
If (W, S) is type Λ and $\boldsymbol{\alpha}$ is a reduced expression with link factorization $\boldsymbol{\alpha}_1 \mid \boldsymbol{\alpha}_2 \mid \cdots \mid \boldsymbol{\alpha}_k$, then $\mathcal{B}(\boldsymbol{\alpha})$ is a partial cube with isometric dimension given by

$$\dim_I(\mathcal{B}(\boldsymbol{\alpha})) = \sum_{i=1}^{k} \operatorname{rank}(\boldsymbol{\alpha}_i).$$

In light of previous theorem about centers determining a link $\boldsymbol{\alpha}$ of rank r, we can define $\Phi_{\boldsymbol{\alpha}} : [\boldsymbol{\alpha}] \to \{0,1\}^r$ via $\Phi_{\boldsymbol{\alpha}}(\boldsymbol{\beta}) = a_1 a_2 \cdots a_r$, where

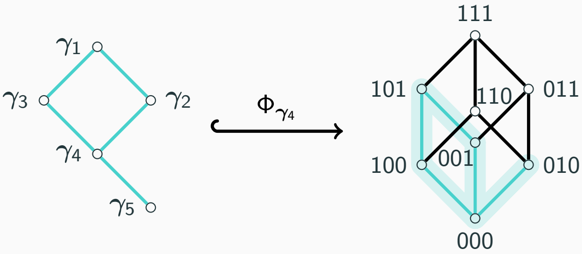
$$a_k = \begin{cases} 0, & \operatorname{sig}_k(\boldsymbol{\beta}) = \operatorname{sig}_k(\boldsymbol{\alpha}) \\ 1, & \text{otherwise.} \end{cases}$$

This map is an isometric embedding of $\mathcal{B}(\boldsymbol{\alpha})$ into Q_r.

Example

Recall the braid chain in type D_4 from earlier:

$$\gamma_1 = \underline{434}1\underline{232},\ \gamma_2 = \underline{343}1\underline{232},\ \gamma_3 = \underline{434}1\underline{323},\ \gamma_4 = \underline{34\overline{3}}\overline{1}\underline{\overline{3}23},\ \gamma_5 = 34\underline{131}23.$$



Some additional results

Theorem

Suppose (W, S) is type Λ and let $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ be links of rank at least one.

- Braid shadows appear once in a geodesic from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$.
- Any two geodesics from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$ utilize same set of braid shadows.
- $d(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \Delta(\mathrm{sig}(\boldsymbol{\alpha}), \mathrm{sig}(\boldsymbol{\beta}))$.
- $\exists\ \boldsymbol{\beta} \in [\boldsymbol{\alpha}]$ that has two non-overlapping braid shadows iff $\mathcal{B}(\boldsymbol{\alpha})$ has a 4-cycle (where opposite edges correspond to same braid shadow).
- If $\mathcal{B}(\boldsymbol{\alpha})$ is a tree, then it is a path.
- Every "primitive cycle" in a braid graph is of length 4.

Open problems & conjectures

Conjectures

For Coxeter systems of type Λ, we conjecture:

- If $\boldsymbol{\alpha}$ is a link, then $\text{diam}(\mathcal{B}(\boldsymbol{\alpha})) = \text{rank}(\boldsymbol{\alpha})$. If true, it follows that that if $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 | \cdots | \boldsymbol{\alpha}_k$ is link factorization, then

$$\text{diam}(\mathcal{B}(\boldsymbol{\alpha})) = \sum_{i=1}^{k} \text{rank}(\boldsymbol{\alpha}_i).$$

- For $\boldsymbol{\alpha}$ a link, there exists a unique diametrical pair $\boldsymbol{\gamma}, \boldsymbol{\mu} \in [\boldsymbol{\alpha}]$.
- If $\boldsymbol{\alpha}$ is a link, then $\mathcal{B}(\boldsymbol{\alpha})$ is underlying graph for Hasse diagram for distributive lattice (diametrical pair are min and max).

Other work to do

- Generalize to arbitrary bond strengths. If all bond strengths odd, fairly certain everything "just works". Even bond strengths?
- Deal with triangle obstruction in Coxeter graph.

Braid graph as Hasse diagram for ranked poset

Construction

- Let $\boldsymbol{\alpha}$ be a link of rank $r \geq 1$.
- Identify diametrical pair of vertices $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ of $\mathcal{B}(\boldsymbol{\alpha})$.
- Elect $\boldsymbol{\mu}$ to be the designated smallest vertex.
- Define $\boldsymbol{\beta} \lessdot \boldsymbol{\sigma}$ if there exists a unique i such that $\text{sig}_i(\boldsymbol{\beta}) \neq \text{sig}_i(\boldsymbol{\sigma})$ and $\Delta(\text{sig}(\boldsymbol{\mu}), \text{sig}(\boldsymbol{\beta})) + 1 = \Delta(\text{sig}(\boldsymbol{\mu}), \text{sig}(\boldsymbol{\sigma}))$.
- $\mathcal{P}(\boldsymbol{\mu}) := ([\boldsymbol{\alpha}], \leq)$ is partial order induced by these covering relations.

Theorem

If (W, S) is of type Λ and $\boldsymbol{\alpha}$ is a link, then

- $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ are adjacent in $\mathcal{B}(\boldsymbol{\alpha})$ iff $\boldsymbol{\beta} \lessdot \boldsymbol{\sigma}$ or $\boldsymbol{\sigma} \lessdot \boldsymbol{\beta}$.
- $\mathcal{P}(\boldsymbol{\mu})$ is ranked by $\Delta(\text{sig}(\boldsymbol{\mu}), \text{sig}(\boldsymbol{\beta}))$
- $\mathcal{B}(\boldsymbol{\alpha})$ is underlying graph for the Hasse diagram of $\mathcal{P}(\boldsymbol{\mu})$.

THANK YOU!