Braid graphs in simply-laced triangle-free Coxeter systems are partial cubes

Part 1

ACGT

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Coxeter Systems

Definition

A Coxeter system consists of a group W (called a Coxeter group) generated by a set S of involutions with presentation

$$W = \langle S \mid s^2 = e, \quad (st)^{m(s,t)} = e \rangle,$$

where $m(s,t) \geq 2$ for $s \neq t$.

Comments

- The elements of S are distinct as group elements.
- $m(s,t)$ is the order of st.

Coxeter Systems

Since s and t are involutions, the relation $(st)^{m(s,t)} = e$ can be rewritten:

$$\left.\begin{array}{lll} m(s,t) = 2 & \Longrightarrow & st = ts \end{array}\right\} \quad \text{commutation relations}$$

$$\left.\begin{array}{lll} m(s,t) = 3 & \Longrightarrow & sts = tst \\ m(s,t) = 4 & \Longrightarrow & stst = tsts \\ & \vdots & \end{array}\right\} \quad \text{braid relations}$$

This allows the replacement

$$\underbrace{sts\cdots}_{m(s,t)} \mapsto \underbrace{tst\cdots}_{m(s,t)}$$

in any word, which is called a commutation move if $m(s,t) = 2$ and a braid move if $m(s,t) \geq 3$.

Coxeter Graphs

Definition

We can encode (W, S) with a unique Coxeter graph Γ having:

- Vertex set $= S$
- $\{s, t\}$ edge labeled with $m(s, t)$ whenever $m(s, t) \geq 3$

Comments

- Typically labels of $m(s, t) = 3$ are omitted.
- Edges correspond to non-commuting pairs of generators.
- Given Γ, we can uniquely reconstruct the corresponding (W, S).

Coxeter Systems of Type A

Example

The Coxeter system of type A_n is defined by the following graph.

Then $W(A_n)$ is subject to:

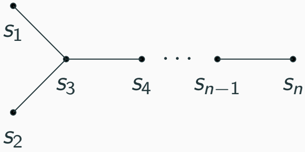
- $s_i^2 = e$ for all i
- $s_i s_j = s_j s_i$ if $|i - j| > 1$
- $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$.

In this case, $W(A_n)$ is isomorphic to the symmetric group S_{n+1} under the correspondence $s_i \mapsto (i, i+1)$.

Coxeter Systems of Type D

Example

The Coxeter system of type D_n is defined by the following graph.



Then $W(D_n)$ is subject to:

- $s_i^2 = e$ for all i
- $s_i s_j = s_j s_i$ when $|i - j| > 1$ and $3 \notin \{i, j\}$
- $s_i s_3 s_i = s_3 s_i s_3$ for $i \in \{1, 2, 4\}$
- $s_i s_j s_i = s_j s_i s_j$ when $|i - j| = 1$ and $i, j \in \{4, 5, \dots, n\}$.

The group $W(D_n)$ is isomorphic to the index 2 subgroup of the group of signed permutations on n letters having an even number of sign changes.

Reduced Expressions & Matsumoto's Theorem

Definition

A word $\alpha = s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ is called an expression for w if it is equal to w when considered as a group element. If m is minimal among all expressions for w, α is a called a reduced expression.

$$\mathcal{R}(w) = \text{set of reduced expressions for } w$$

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Matsumoto's Theorem

Any two reduced expressions for $w \in W$ differ by a sequence of commutation & braid moves.

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Matsumoto's Theorem

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Definition

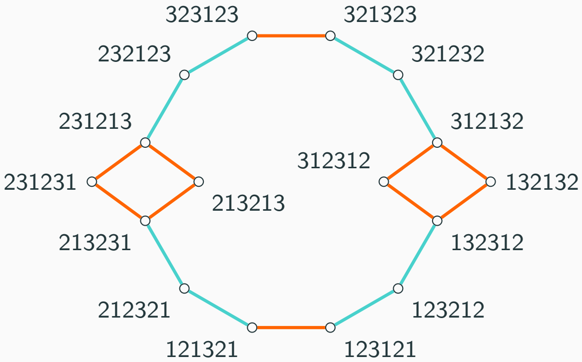
For $w \in W$, define the Matsumoto graph $\mathcal{M}(w)$ via:

- Vertex set $= \mathcal{R}(w)$
- $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ edge iff $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are related via a single commutation or braid move

Matsumoto Graph

Example

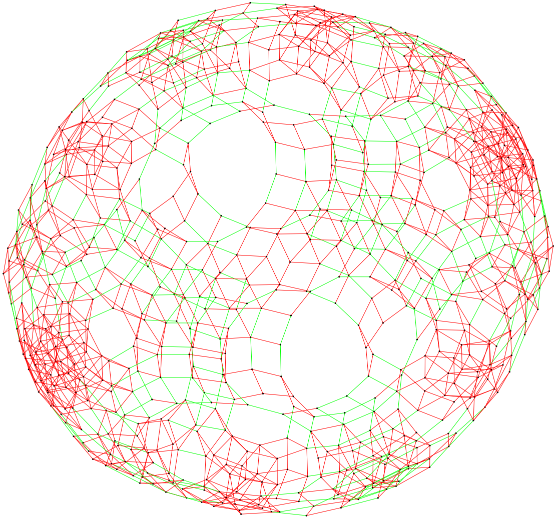
Consider the reduced expression $\boldsymbol{\alpha} = 121321$ for $w \in W(A_3)$. Then $\mathcal{M}(w)$ is as follows:



Matsumoto Graph

Example

Here is the Matsumoto graph for the longest element in type A_4.



Braid Equivalence & Braid Graphs

Definition

If $\alpha, \beta \in \mathcal{R}(w)$, then α and β are braid equivalent iff α and β are related by a sequence of braid moves.

Comments

- Braid equivalence is an equivalence relation.
- Equivalence classes are called braid classes, denoted $[\alpha]$.

Braid Equivalence & Braid Graphs

Definition

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- Equivalence classes are called braid classes, denoted $[\boldsymbol{\alpha}]$.

Definition

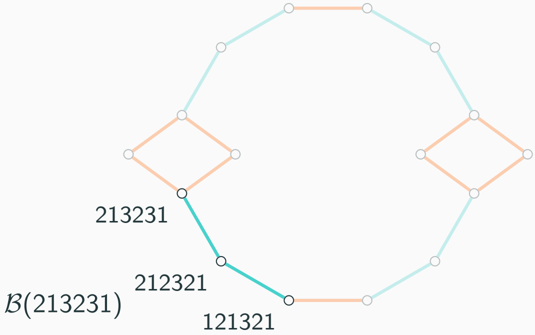
We can encode a braid class $[\boldsymbol{\alpha}]$ in a braid graph, denoted $\mathcal{B}(\boldsymbol{\alpha})$:

- Vertex set $= [\boldsymbol{\alpha}]$
- $\{\boldsymbol{\gamma}, \boldsymbol{\beta}\}$ edge iff $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ are related via a single braid move

Braid graphs are the maximal turquoise connected components in the Matsumoto graph. Not to be confused with contracting the braid edges of a Matsumoto graph.

Braid Graphs

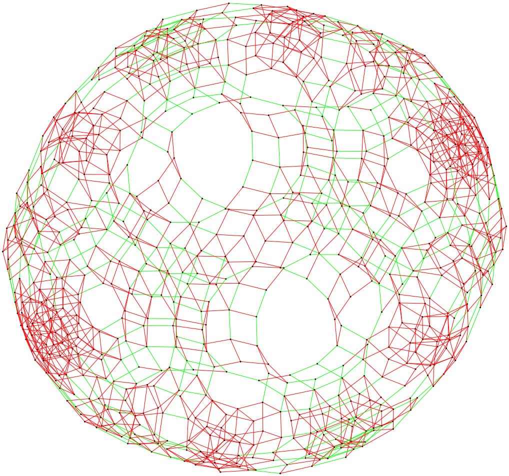
Example



Braid Graphs

Example

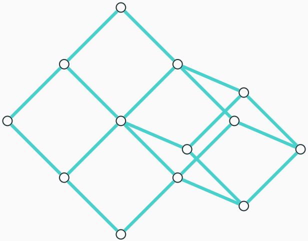
Each of the maximal green connected components in the following Matsumoto graph is a braid graph corresponding to a braid class.



Braid Graphs

Example

Consider the reduced expression $\boldsymbol{\alpha} = 31323431323$ for some $w \in W(D_4)$. Then $\mathcal{B}(\boldsymbol{\alpha})$ is as follows, where $\boldsymbol{\alpha}$ is the vertex of degree 5.



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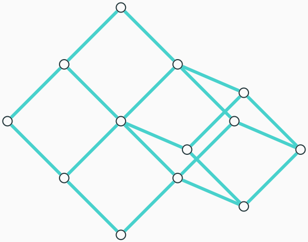

Big Picture Goal

Characterize the structure of braid classes/graphs with an aim at understanding the relationship among the reduced expressions for a group element.

Links & Braid Chains

Definition

Suppose α is a reduced expression for $w \in W$ consisting of m letters. Loosely speaking, α is link if there is a sequence of overlapping braid moves that "cover" the positions $1, 2, \ldots, m$. If α is a link, then the corresponding braid class $[\alpha]$ is called a braid chain.

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Example

Consider the reduced expression $\alpha = 343546576$ for some $w \in W(A_7)$.

343546576   **43454**6576   43**54565**76   4354**65676**   435465**767**

In this example, every reduced expression is a link and the braid class is a braid chain.

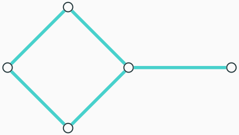
Links & Braid Chains

Example

Let $\alpha = 3134323$ be a reduced expression for some $w \in W(D_4)$.

$$\overline{31\underline{3}}\underline{4}\overline{\underline{3}23}$$

Then α is a link and $[\alpha]$ is a braid chain. The corresponding braid graph is as follows, where α is the vertex of degree 3.



Links & Braid Chains

Example

Now, let $\alpha = 1213243676$ be a reduced expression for some $w \in W(A_7)$.
It turns out that α is not a link, but rather a product of two links.

$$1213243 \mid 676$$

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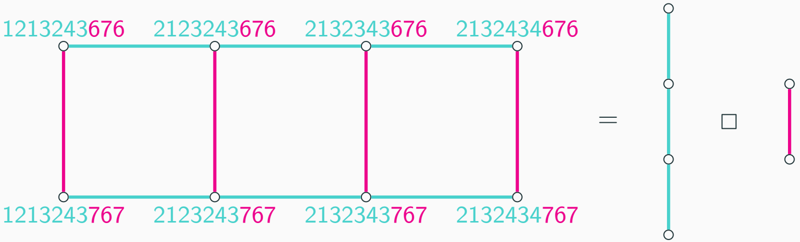

Braid Link Factorizations

Comments

- Every reduced expression factors uniquely into maximal links, called a braid link factorization.
- Describing the maximal links and their corresponding braid chains is tricky business!
- We have a nice characterization for triangle-free simply-laced Coxeter systems.

Braid Graphs for Braid Factorizations

Theorem

If $\boldsymbol{\alpha}$ is a reduced expression for $w \in W$ having braid link factorization

$$\boldsymbol{\alpha} = \boldsymbol{\beta}_1 \mid \boldsymbol{\beta}_1 \mid \cdots \mid \boldsymbol{\beta}_m,$$

then $\mathcal{B}(\boldsymbol{\alpha})$ is the box product of the braid graphs for each $\boldsymbol{\beta}_i$.

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Comment

- The upshot is that if you want to understand the structure of braid graphs, you must first characterize the braid graphs for links.
- We've classified the braid graphs for links in types A_n and D_n (and others at least "in my head").
- In the case of type A_n, links have odd length and the corresponding braid graphs are paths.

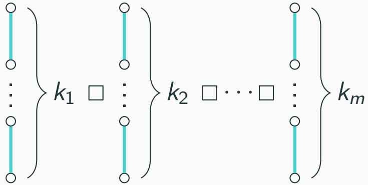
Theorem (Fisher et al. $\to$ Bidari & Ernst)

If $\boldsymbol{\alpha}$ is a reduced expression for $w \in W(A_n)$ having braid link factorization

$$\boldsymbol{\alpha} = \boldsymbol{\beta}_1 \mid \boldsymbol{\beta}_2 \mid \cdots \mid \boldsymbol{\beta}_m$$

such that each factor has $2k_i - 1$ generators, then

$\mathcal{B}(\boldsymbol{\alpha}) =$

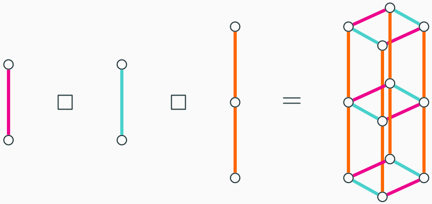


Example

Consider the following braid link factorization for a reduced expression for an element in $W(A_7)$.

$$\boldsymbol{\alpha} = \mathbf{121} \mid \mathbf{434} \mid \mathbf{65676}$$

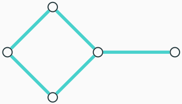
The resulting braid graph is shown below:



Fibonacci Links in Type D

Consider the Coxeter system of type D_4. Let $\{a, b, c\} = \{1, 2, 4\}$. Every reduced expression that is braid equivalent to one of the following is called a Fibonacci link (in type D_4). The corresponding braid graph is depicted on the right.

$3a3b3c3$



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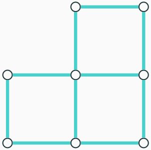

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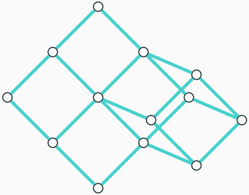

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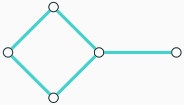

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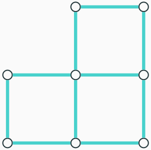

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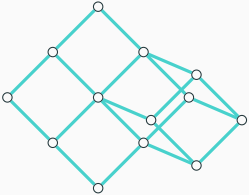

Each one of the graphs above corresponds to a Fibonacci cube graph!

Classification of Braid Graphs for Links in Type D

Theorem

In type D_n, every link is braid equivalent to either a "type A" link or a "type A extension" of a Fibonacci link. As a consequence, braid graphs for links in type D_n are either paths or "type A extensions" of braid graphs for Fibonacci links.

Choices for a, b, c determine whether we can extend; need 343 on an end.

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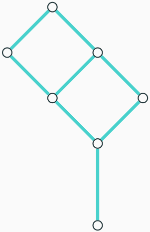
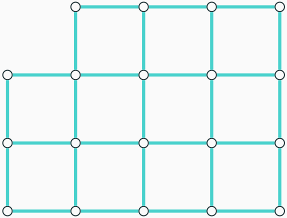


453431323



45343132343 5465

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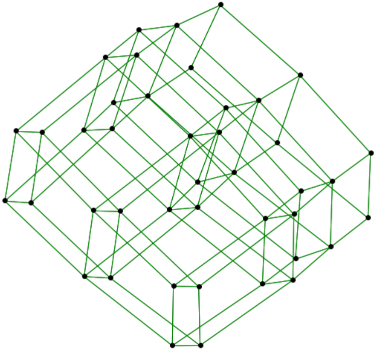

$\alpha = 453431323 \mid 56576 \mid 898$