Impartial geodetic convexity achievement & avoidance games on graphs

Combinatorial Game Theory Colloquium IV

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January 25, 2023

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Partial support from The Institute for Computational and Experimental Research in Mathematics (ICERM)

Graph Theory

- We assume collection of vertices V is nonempty and finite.
- A geodesic of a graph is a shortest path between two vertices. The geodetic closure $I[P]$ of a subset $P \subseteq V$ consists of the vertices along the geodesics connecting two vertices in P.
- A subset $P \subseteq V$ is called (geodetically) convex if it contains all vertices along the geodesics connecting two vertices of P.
- The convex hull of P is defined via

  $$[P] := \bigcap \{K \mid P \subseteq K, K \text{ is convex}\}$$

  and is the smallest convex set containing P.
- We say that a subset P of vertices is generating if $[P] = V$.

Geodetic Closure vs Convex Hull

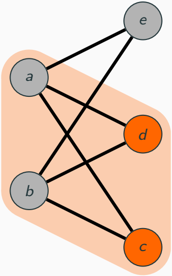
Comments

- Despite the name, geodetic closure is not necessarily a closure operator because it may not be idempotent. To make a closure operator, we need to iterate the geodetic closure function until the result stabilizes.
- Convex hull is this closure operator.

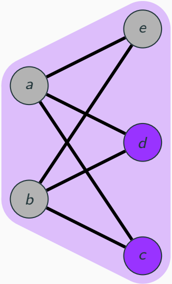
Example

Consider the complete bipartite graph $K_{2,3}$.

$I[\{c, d\}]$



$[\{c, d\}]$

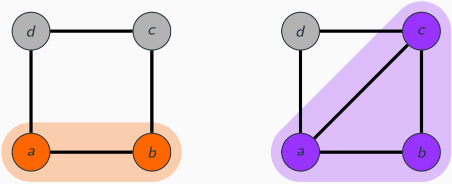


Definition

The family of maximal nongenerating sets of a graph G is denoted by $\mathcal{N}(G)$. That is, $\mathcal{N}(G) := \{N \subseteq V \mid [N] \neq V \text{ but for all } v \notin N, [N \cup \{v\}] = V\}$.

Example

Consider the cycle graph C_4 and the diamond graph G.



C_4 G

The maximal nongenerating subsets of C_4 are $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, $\{a, d\}$. On the other hand, the maximal nongenerating sets of the diamond graph are $\{a, b, c\}$ and $\{a, c, d\}$.

Game Definitions

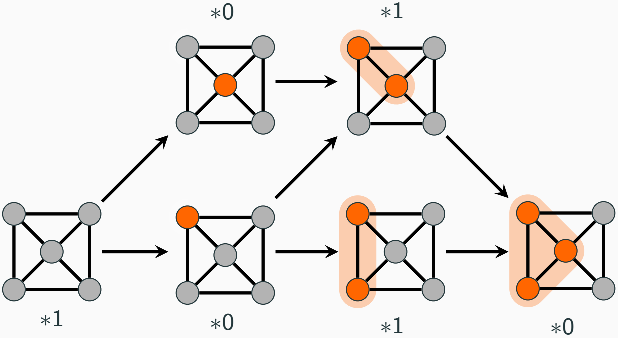
Definition

For each of the games, we play on a graph $G = (V, E)$. Two players take turns selecting previously unselected vertices until certain conditions are met.

- For the achievement game generate $\text{GEN}(G)$, the game ends as soon as $[P] = V$. That is, the player who generates the whole vertex set first wins.
- For the avoidance game do not generate $\text{DNG}(G)$, all positions P must satisfy $[P] \neq V$. The player who cannot select a vertex without generating the vertex set loses.

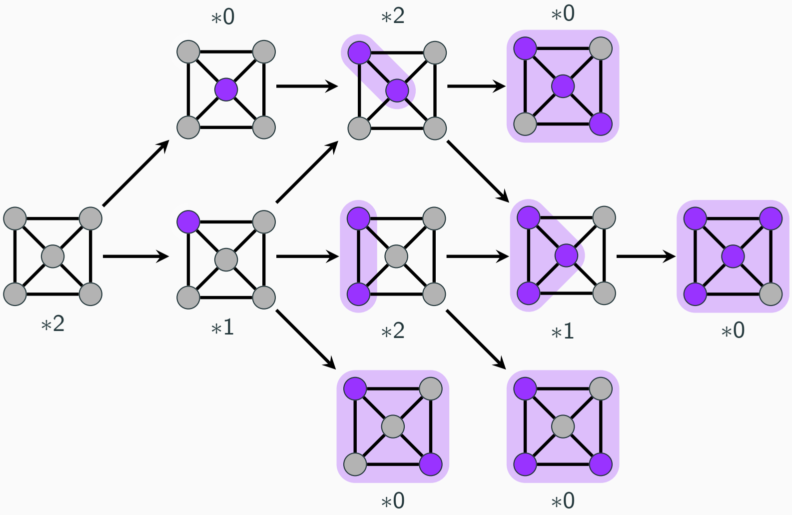
Example

Consider the wheel graph W_5. Below is a “representative” game digraph for DNG(W_5). *Note:* Positions can never contain antipodal “rim” vertices.



Example

Below is a "representative" game digraph for GEN(W_5).



Comments

Similar games have been considered by several authors, including Buckley/Harary, Fraenkel/Harary, Necascova, Haynes/Henning/Tiller, and Wang. These variations differ in at least one of the following:

- The collection of vertices generated by the selected vertices corresponds to the geodetic closure as opposed to the convex hull. (Buckley/Harary)
- The generated vertices of the selected vertices are not available as moves.

The games we study are a generalization of the achievement and avoidance games played on groups introduced by Anderson/Harary and extensively studied by Benesh/Ernst/Sieben.

Comments

The games DNG(G) and GEN(G) are completely determined by $\mathcal{N}(G)$.

- The set of terminal positions of DNG(G) is $\mathcal{N}(G)$.
- A subset $P \subseteq V$ is a position of GEN(G) if and only if $P \setminus \{v\} \subseteq N$ for some $v \in V$ and $N \in \mathcal{N}(G)$.

The following theorem quickly handles the determination of the nim-number for DNG(G) for several families of graphs.

Theorem (BEMSS)

If G is a graph and every element of $\mathcal{N}(G)$ has the same parity $r \in \{0, 1\}$, then the nim-number of DNG(G) is r.

Theorem (BEMSS)

For the complete graph K_n, we have:

- $\mathcal{N}(K_n) = \{V \setminus \{v\} \mid v \in V\}$.
- $\text{nim}(\text{DNG}(K_n)) = \text{pty}(n-1)$.

  Proof. This follows from "This one is easy" since every position of $\mathcal{N}(K_n)$ has the same parity. □
- $\text{nim}(\text{GEN}(K_n)) = \text{pty}(n)$.

  Proof. The only way to generate V is to select each vertex. If n is even, the second player wins by random play. If n is odd, the second player wins $\text{GEN}(K_n) + *1$ again by random play. □

Theorem (BEMSS)

If T is a tree with set of leaves of L, then we have:

- $\mathcal{N}(T) = \{\{l\}^c \mid l \in L\}$.
- $\text{nim}(\text{DNG}(T)) = \text{pty}(|V|-1)$.

  Proof. Again, this follows from "This one is easy" since every position of $\mathcal{N}(K_n)$ has the same parity. □
- $\text{nim}(\text{GEN}(T)) = \text{pty}(V)$.

  Proof. One approach is to use structural induction on the diagram that results from structure equivalence. □

Theorem (BEMSS)

For the cycle graph C_n ($n \geq 3$), assume $V = \mathbb{Z}_n$ and $E = \{\{i, i+1\} \mid i \in V\}$.

- $\mathcal{N}(C_n) = \begin{cases} \{\{i+1, \ldots, i+(n+1)/2\} \mid i \in V\}, & \text{if } n \text{ odd} \\ \{\{i+1, \ldots, i+n/2\} \mid i \in V\}, & \text{if } n \text{ even} \end{cases}.$
- $\text{nim}(\text{DNG}(C_n)) = \begin{cases} 1, & \text{if } n \equiv_4 1, 2 \\ 0, & \text{if } n \equiv_4 3, 0. \end{cases}$

  Proof. Surprise! ... "This one is easy" (some thought required to determine parity). □
- $\text{nim}(\text{GEN}(C_n)) = \text{pty}(n)$.

  Proof. If n is even, then 2nd player wins in 2nd move by selecting the antipodal vertex. If n is odd, then 1st player wins on 3rd move by selecting a vertex in the "middle" of the larger group of unselected vertices. □

Hypercube Graphs

Theorem (BEMSS)

For the hypercube graph Q_n (binary strings vertices connected by an edge exactly when they differ by a single digit), we have:

- For $n \geq 2$, $\mathcal{N}(Q_n)$ is collection of sets consisting of vertices agreeing on a fixed entry.
- $\text{nim}(\text{DNG}(Q_n)) = 0$.

  Proof. Note that $Q_1 = K_1$, so the result follows from earlier theorem. For $n \geq 2$, every set in $\mathcal{N}(Q_n)$ has size 2^{n-1}, so the result follows from "This one is easy".
- $\text{nim}(\text{GEN}(Q_n)) = 0$.

  Proof. The 2nd player wins by selecting the antipodal vertex to the choice of 1st player, and every antipodal pair forms a minimal generating set.

Theorem (BEMSS)

Consider the complete bipartite graph $K_{m,n}$ where $n \geq m \geq 2$ with the set V of vertices partitioned into $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$. Then:

- $\mathcal{N}(K_{m,n}) = \{\{a_i, b_j\} \mid a_i \in A, b_j \in B\}$.
- $\text{nim}(\text{DNG}(K_{m,n})) = 0$.

  Proof. "This one is easy" since every position of $\mathcal{N}(K_{m,n})$ has size two.
- $\text{nim}(\text{GEN}(K_{m,n})) = 0$.

  Proof. The 2nd player wins on their first turn by selecting a vertex in the same part as the 1st player.

Wheel Graphs

Theorem (BEMSS)

We define the wheel graph W_n ($n \geq 5$) to be graph with $V = \{v_1, \ldots, v_{n-1}, c\}$, where c is the center and v_i is adjacent to v_{i+1} (considered modulo $n-1$).

- $\mathcal{N}(W_n) =$ complements of sets containing 2 neighboring "rim" vertices.
- $\text{nim}(\text{DNG}(W_n)) = \text{pty}(n)$.

  Proof. Each set in $\mathcal{N}(W_n)$ has size $n-2$, so ... "This one is easy".
- $\text{nim}(\text{GEN}(W_n)) = \begin{cases} 2, & n = 5 \\ \text{pty}(n), & n \geq 6. \end{cases}$

  Proof. The case involving $n = 5$ handled separately. When $n \geq 6$ and even, not hard to argue that 2nd player has winning strategy. When $n \geq 7$ and odd, 2nd player has a winning strategy in the game $\text{GEN}(W_n) + *1$ using a pairing strategy until near end of game (complicated case analysis).

But wait, there's more!

Comments

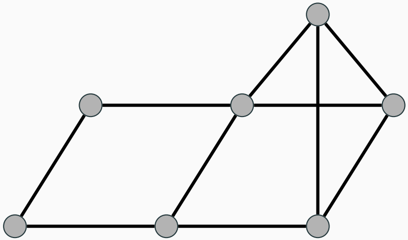
- We have obtained general results concerning maximal nongenerating sets for disjoint unions of graphs, 1-clique sums of graphs, and products of graphs. Except in some specialized circumstances, there do not seem to be straightforward results concerning nim-numbers for any of these situations.
- We have obtained nim-numbers for generalized windmill graphs, complete multipartite graphs.
- In many instances (e.g., complete graphs, trees, cycles, wheel graphs), geodetic closure is the same as convex hull of a set. In these cases, we have also settled the Buckley/Harary versions of the game. Not true for hypercube graphs and complete bipartite graphs.
- We have also obtained analogous results for the complementary "removing" games Terminate and Do Not Terminate.

Conjecture

We conjecture that the spectrum of nim-numbers for GEN and DNG is $\mathbb{N} \cup \{0\}$. We have examples of graphs that exhibit $*0, *1, *2, *3, *4, *5, *6, *7$.

Example

If G is the following graph, then $\mathrm{DNG}(G) = *5$.



Frattini Subset

Recall that the Frattini subgroup of a group G is the intersection of all maximal subgroups of G. We make the analogous definition in terms of maximal nongenerating sets of a graph

Definition

We define the Frattini subset of a graph G via $\Phi(G) := \bigcap \mathcal{N}(G)$.

The Frattini subgroup is equivalently defined as the collection of nongenerators of the group. Indeed, we have the analogous theorem for graphs.

Definition

A vertex v is called a nongenerator if for all subsets S of vertices, $[S] = V$ implies $[S \setminus \{v\}] = V$.

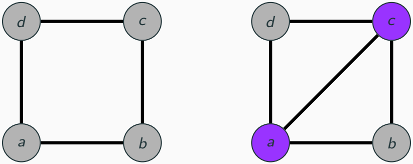
Theorem (BEMSS)

The set of nongenerators of a graph G is the Frattini subset $\Phi(G)$.

Frattini Subset (continued)

Example

Recall that the maximal nongenerating subsets of C_4 and the diamond graph are $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, $\{a, d\}$ and $\{a, b, c\}$, $\{a, c, d\}$, respectively.



C_4 G

Hence the corresponding Frattini subsets are $\emptyset$ and $\{a, c\}$, respectively.

Open Problem

Is the Frattini subset related to known graph-theoretic concepts? Possibly related to "minimal eccentricity approximating spanning trees"???

Frattini Subset (continued)

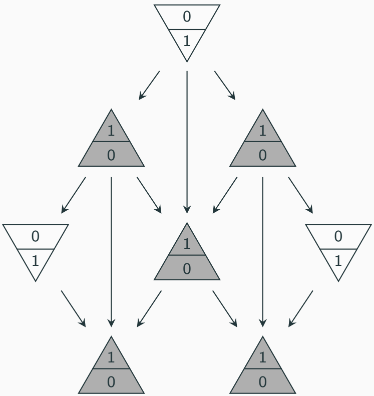
In some more complicated situations (e.g., 2-dimensional lattice graphs), our method of attack involves simplifying game digraph by partitioning the collection of positions into so-called structure classes where both the option relationship between positions and the corresponding nim-numbers are compatible with structure equivalence according to parity.

Theorem (BEMSS)

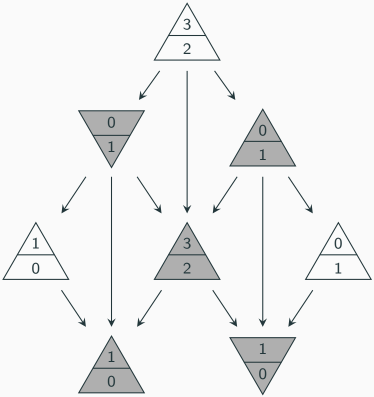
- For both games, the starting position $\emptyset$ is always contained in structure class containing the Frattini subset $\Phi(G)$.
- In each case, the nim-number of the game equals the nim-number of the even-parity positions contained in the structure class containing $\Phi(G)$.

Example

Below are the "simplified" structure diagrams for two cases of DNG($P_n \square P_m$) .



(i) n and m odd



(ii) pty(n) $\neq$ pty(m) & neither is 2

Theorem (BEMSS)

For the 2-dimensional lattice graph $P_n \square P_m$, we have:

- The maximal nongenerating sets for $P_n \square P_m$ correspond to the complement of the vertices lying along one of the 4 exterior sides of the grid.
- $\Phi(P_n \square P_m)$ is the "interior" of the grid.
- $\mathrm{nim}(\mathrm{DNG}(P_n \square P_m)) = \begin{cases} 0, & \text{if } \mathrm{pty}(n) = \mathrm{pty}(m) \text{ or } \min\{m, n\} = 2 \\ 2, & \text{otherwise.} \end{cases}$
- $\mathrm{nim}(\mathrm{GEN}(P_n \square P_m)) = \begin{cases} 0, & \text{if } n \text{ or } m \text{ is even} \\ 1, & \text{if } n \text{ and } m \text{ are odd.} \end{cases}$
- Proofs for both involve structural induction.