Diagram algebras as combinatorial tools for exploring Kazhdan–Lusztig theory

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Definition

A Coxeter group is a group W with a distinguished set of generating involutions S having presentation

$$W = \langle S : s^2 = 1, (st)^{m(s,t)} = 1 \rangle,$$

where $m(s,t) \geq 2$ for $s \neq t$ and $m(s,s) = 1$.

Comment

Since s and t are involutions, the relation $(st)^{m(s,t)} = 1$ can be rewritten as

$$\begin{array}{lcrl}
m(s,t) = 2 & \Longrightarrow & st = ts & \big\} \quad \text{short braid relations} \\[1em]
m(s,t) = 3 & \Longrightarrow & sts = tst & \multirow{3}{*}{$\left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} \quad \text{long braid relations}$} \\
m(s,t) = 4 & \Longrightarrow & stst = tsts & \\
 & \vdots & &
\end{array}$$

Definition

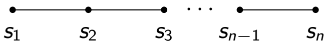
We can encode (W, S) with a unique Coxeter graph Γ having:

- vertex set S;
- edges $\{s, t\}$ labeled $m(s, t)$ whenever $m(s, t) \geq 3$
- if $m(s, t) = 3$, we omit label.

Comments

- If s and t are not connected in Γ, then s and t commute.
- Given Γ, we can uniquely reconstruct the corresponding (W, S).

Coxeter groups of type A_n ($n \geq 1$) are defined by:



Then $W(A_n)$ is generated by $\{s_1, s_2, \cdots, s_n\}$ and is subject to defining relations

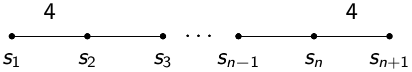
1. $s_i^2 = 1$ for all i,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$,
3. $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$.

$W(A_n)$ is isomorphic to the symmetric group, S_{n+1}, under the correspondence

$$s_i \mapsto (i \; i+1),$$

where $(i \; i+1)$ is the adjacent transposition exchanging i and $i+1$.

Coxeter groups of type $\widetilde{C}_n$ ($n \geq 2$) are defined by:



Here, we see that $W(\widetilde{C}_n)$ is generated by $\{s_1, \cdots, s_{n+1}\}$ and is subject to defining relations

1. $s_i^2 = 1$ for all i,
2. $s_i s_j = s_j s_i$ if $|i - j| > 1$,
3. $s_i s_j s_i = s_j s_i s_j$ if $|i - j| = 1$ and $1 < i, j < n + 1$,
4. $s_i s_j s_i s_j = s_j s_i s_j s_i$ if $\{i, j\} = \{1, 2\}$ or $\{n, n + 1\}$.

$W(\widetilde{C}_n)$ is an infinite group.

Definition

A word $s_{x_1} s_{x_2} \cdots s_{x_m} \in S^*$ is called an expression for $w \in W$ if it is equal to w when considered as a group element.

If m is minimal, it is a reduced expression, and the length of w is $\ell(w) := m$.

Given $w \in W$, if we wish to emphasize a fixed, possibly reduced, expression for w, we represent it as

$$\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_m}.$$

Theorem (Matsumoto)

Any two reduced expressions for $w \in W$ differ by a sequence of braid relations.

Example

Let $w \in W(A_3)$ with expression $\overline{w} = s_2 s_1 s_2 s_3 s_1$. Since $s_2 s_1 s_2 = s_1 s_2 s_1$, $s_1 s_3 = s_3 s_1$, and $s_1^2 = 1$, we see that

$$s_2 s_1 s_2 s_3 s_1 = s_1 s_2 s_1 s_3 s_1 = s_1 s_2 s_1 s_1 s_3 = s_1 s_2 s_3.$$

This shows that $\overline{w}$ is *not* reduced. However, it is true (since there are no relations left to apply) that $s_1 s_2 s_3$ is a reduced expression for w, so that $\ell(w) = 3$.

Definition

We say that $w \in W$ is fully commutative (FC) if any two reduced expressions for w can be transformed into each other via iterated commutations. The set of FC elements of W is denoted by $\mathrm{FC}(W)$.

Theorem (Stembridge)

$w \in \mathrm{FC}(W)$ *iff no reduced expression for* w *contains a long braid.*

Comments

The FC elements of $W(\widetilde{C}_n)$ are precisely those that avoid the following consecutive subexpressions:

1. $s_i s_j s_i$ for $|i-j| = 1$ and $1 < i, j < n+1$,
2. $s_i s_j s_i s_j$ for $\{i, j\} = \{1, 2\}$ or $\{n, n+1\}$.

It follows from work of Stembridge that $W(\widetilde{C}_n)$ contains an infinite number of FC elements. There are examples of infinite Coxeter groups that contain a finite number of FC elements (e.g., type E_n for $n \geq 9$).

Example

Let $w \in W(\widetilde{C}_3)$ have reduced expression $\overline{w} = s_1 s_3 s_2 s_1 s_2$. Since s_1 and s_3 commute, we can write

$$w = s_1 s_3 s_2 s_1 s_2 = s_3 s_1 s_2 s_1 s_2.$$

This shows that w has a reduced expression containing $s_1 s_2 s_1 s_2$ as a consecutive subexpression, which implies that w is not FC.

Now, let $w' \in W(\widetilde{C}_3)$ have reduced expression $\overline{w}' = s_1 s_2 s_1 s_3 s_2$. Then we will never be able to rewrite w' to produce one of the illegal consecutive subexpressions since the only relation we can apply is

$$s_1 s_3 \to s_3 s_1$$

which does not provide an opportunity to apply any additional relations. So, w' is FC.

Given a Coxeter group W, the associated Hecke algebra is an algebra with a basis indexed by the elements of W and relations that deform the relations of W by a parameter q. (If we set q to 1, we recover the group algebra of W.) More specifically:

Definition

The associative $\mathbb{Z}[q, q^{-1}]$-algebra $\mathcal{H}_q(W)$ is the free module on the set $\{T_w : w \in W\}$ that satisfies

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } \ell(sw) > \ell(w), \\ qT_{sw} + (q-1)T_w, & \text{otherwise.} \end{cases}$$

After "extending" scalars so that $v^2 = q$, we obtain the Hecke algebra $\mathcal{H}(W)$.

If $\overline{w} = s_{x_1} s_{x_2} \cdots s_{x_m}$ is a reduced expression for $w \in W$, then

$$T_w = T_{s_{x_1}} T_{s_{x_2}} \cdots T_{s_{x_m}}.$$

Skipping most of the details...

Theorem (Kazhdan, Lusztig)

There is a unique basis $\{C'_w : w \in W\}$ (canonical basis) for $\mathcal{H}(W)$ with "important and subtle properties".

Definition

The Kazhdan–Lusztig polynomials occur as follows. If

$$C'_w = \sum_{y \leq w} P^*_{y,w} \widetilde{T_y},$$

where $\leq$ is the Bruhat order on the Coxeter group W, then

$$P_{y,w} := v^{\ell(w)-\ell(y)} P^*_{y,w}.$$

Properties

- $P_{y,w} = 0$ unless $y \leq w$ (Bruhat order).
- $P_{w,w} = 1$ for all $w \in W$.
- $P_{y,w} \in \mathbb{Z}_{\geq 0}[q]$ . . . deep!
- If $P_{y,w} \neq 0$, then $\deg(P_{y,w}) \leq \frac{1}{2}(\ell(w) - \ell(y) - 1)$.
- We write $\mu(y, w) \in \mathbb{Z}$ for the coefficient of $q^{1/2(\ell(w)-\ell(y)-1)}$ in $P_{y,w}$.
- There is a (terrifying looking!) recursive formula

$$P_{x,w} = q^{1-c} P_{sx,v} + q^c P_{x,v} - \sum_{z \prec v, sz < z} \mu(z, w) q^{1/2(\ell(w)-\ell(z)-1)} P_{x,z},$$

where $sw = v < w$ and $c = \begin{cases} 0, & \text{if } x < sx \\ 1, & \text{otherwise.} \end{cases}$

Comments

- K–L polynomials have applications to the representation theory of semisimple algebraic groups, Verma modules, algebraic geometry and topology of Schubert varieties, etc.
- There is natural basis indexed by the elements of W for: $\{T_w\}$.
- There is this another really nice basis that we like better: $\{C'_w\}$.
- The K–L polynomials essentially occur as the entries in the change of basis matrix from one basis to the other.
- The μ-values occur as the coefficients on the highest degree term in the corresponding K–L polynomial.
- Unfortunately, computing the polynomials efficiently quickly becomes difficult, even in finite groups of moderate size.
- There are quite a few people (like myself) trying to find non-recursive ways to compute μ-values.

Definition

Let W be a Coxeter group with graph Γ. Define $J(W)$ be the two-sided ideal of $\mathcal{H}(W)$ generated by

$$\sum_{w \in \langle s, s' \rangle} T_w,$$

where (s, s') runs over all pairs of of elements of S with $3 \leq m(s, s') < \infty$, and $\langle s, s' \rangle$ is the (parabolic) subgroup generated by s and s'. We define the (generalized) Temperley–Lieb algebra, $\mathrm{TL}(\Gamma)$, to be the quotient algebra $\mathcal{H}(W)/J(W)$.

Theorem (Graham)

Let t_w denote the image of T_w in the quotient. Then $\{t_w : w \in \mathrm{FC}(W)\}$ is a basis for $\mathrm{TL}(\Gamma)$.

Definition

For each $s_i \in S$, define $b_i = v^{-1} t_{s_i} + v^{-1}$. If $w \in \mathrm{FC}(W)$ has reduced expression $\overline{w} = s_{x_1} \cdots s_{x_m}$, define

$$b_w = b_{x_1} \cdots b_{x_m}.$$

Theorem (Graham)

The set $\{b_w : w \in \mathrm{FC}(W)\}$ *(monomial basis) forms a basis for* $\mathrm{TL}(\Gamma)$.

Theorem (Graham)

$\mathrm{TL}(\widetilde{C}_n)$ *is generated (as unital algebra) by* $b_1, b_2, \ldots, b_{n+1}$ *with defining relations*

1. $b_i^2 = \delta b_i$ *for all* i, *where* $\delta = v + v^{-1}$
2. $b_i b_j = b_j b_i$ *if* $|i - j| > 1$,
3. $b_i b_j b_i = b_i$ *if* $|i - j| = 1$ *and* $1 < i, j < n + 1$,
4. $b_i b_j b_i b_j = 2 b_i b_j$ *if* $\{i, j\} = \{1, 2\}$ *or* $\{n, n + 1\}$.

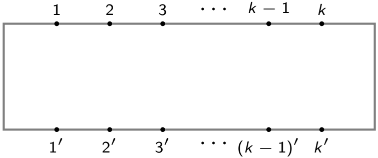
$\mathrm{TL}(A_n)$ *is generated by* $b_2, \ldots, b_n$ *together with the corresponding relations.*

Motivation

Loosely speaking, $\mathrm{TL}(\Gamma)$ retains some of the relevant structure of $\mathcal{H}(W)$, yet is small enough that the computation of the μ-values of the K–L polynomials is often simpler.

Definition

A standard k-box is a rectangle with $2k$ nodes, labeled as follows:

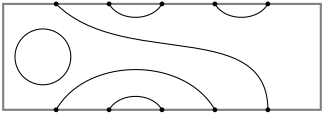


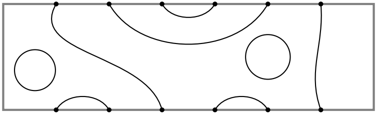
A pseudo k-diagram (or an ordinary Temperley-Lieb pseudo diagram) consists of a finite number of disjoint (planar) edges embedded in and disjoint from the standard k-box such that

1. edges may be closed (isotopic to circles), but not if their endpoints coincide with the nodes of the box;
2. the nodes of the box are the endpoints of curves, which meet the box transversely.

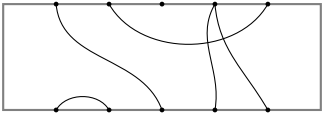
Example

Here are two examples of concrete pseudo diagrams.





Here is an example that is not a concrete pseudo diagram.



Definition

The (type A) Temperley–Lieb diagram algebra, denoted by $\mathbb{D}\mathrm{TL}(A_n)$, is the free $\mathbb{Z}[\delta]$-module with basis consisting of the pseudo $(n+1)$-diagrams having no loops.

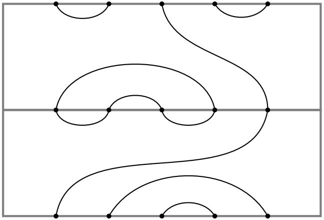
To calculate the product dd' identify the S-face of d with the N-face' of d' and then multiply by a factor of δ for each resulting loop and then discard the loop.

Theorem

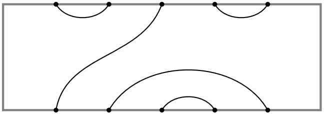
$\mathbb{D}TL(A_n)$ is an associative $\mathbb{Z}[\delta]$-algebra having the loop-free pseudo $(n+1)$-diagrams as a basis.

Example

Multiplication of two concrete pseudo 5-diagrams.

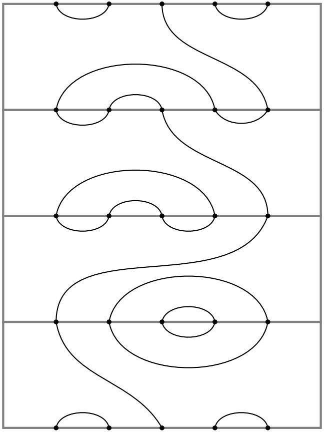


$= \delta$

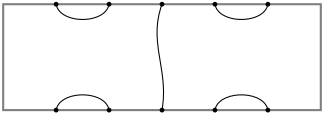


Example

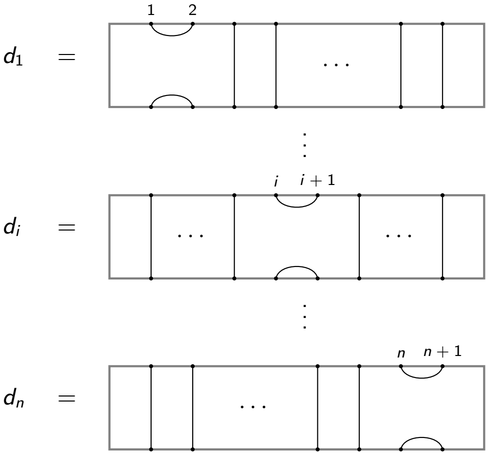
And here's another example.



$= \delta^3$



Now, we define the set of simple diagrams, which turn out to form a generating set for $\mathbb{D}\mathrm{TL}(A_n)$.



Theorem

The $\mathbb{Z}[\delta]$-algebra homomorphism $\theta : \mathsf{TL}(A_n) \to \mathbb{D}TL(A_n)$ determined by

$$\theta(b_i) = d_i$$

is an algebra isomorphism. Moreover, the loop-free pseudo diagrams are in bijection with the monomial basis elements of $\mathsf{TL}(A_n)$.

Theorem (R.M. Green)

If $y, w \in W(A_n)$ with both FC, then $\mu(y, w)$ can be computed (non-recursively) as follows.

1. *Draw diagrams for d_y and $d_{w^{-1}}$.*
2. *Multiply d_y times $d_{w^{-1}}$. Do not replace any closed loops with δ.*
3. *Connect point i in N-face to point i' in S-face (without intersections).*

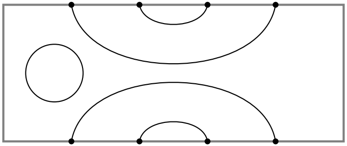
If this forms n closed loops, then $\mu(y, w) = 1$, and otherwise, $\mu(y, w) = 0$.

Example

Let $\overline{y} = s_2$ and $\overline{w} = s_2 s_1 s_3 s_2$ be reduced expressions for y and w, respectively, in $W(A_3)$. Note that both y and w are FC. We see that $w^{-1} = s_2 s_3 s_1 s_2$. Then

$$d_y d_{w^{-1}} = d_2 d_2 d_3 d_1 d_2,$$

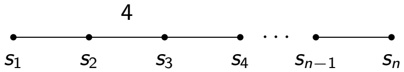
which yields the following pseudo diagram:



If we "wrap up" this diagram, we see that there are 3 loops. Therefore, by the previous theorem, $\mu(y, w) = 1$.

Comments

- What we are really doing when we "wrap up" $d_y d_{w^{-1}}$ is defining a Jones-type trace on a quotient of the Hecke algebra.
- When this type of trace function is known to exist, we can use it to compute $\mu(y, w)$ for $y, w \in \mathrm{FC}(W)$.
- At this point, only when we have a diagrammatic representation have we been able to define the necessary trace to non-recursively compute μ-values.
- Current state of affairs: we can do this when Γ is of type A, B, D, H, E, or $\widetilde{A}$. (See papers by R.M. Green.)
- Coming soon: type $\widetilde{C}$.
- Elusive: type F.



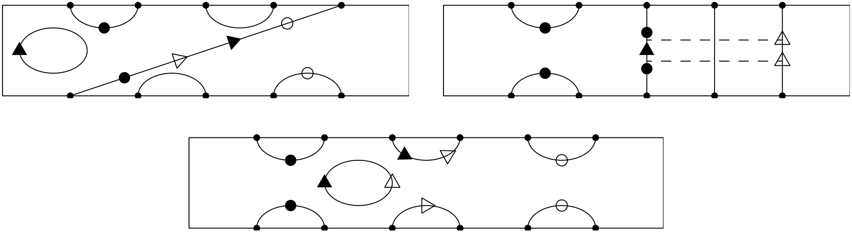
Now, we will *briefly* describe a representation of $\mathrm{TL}(\widetilde{C}_n)$ that involves decorated diagrams. For our decoration set, we take $\Omega = \{\bullet, \blacktriangle, \circ, \vartriangle\}$.

Definition

An LR-decorated pseudo diagram is any Ω-decorated concrete diagram subject to "a few constraints" about how we place decorations on the edges.

Example

Here are some examples of LR-decorated pseudo diagrams.

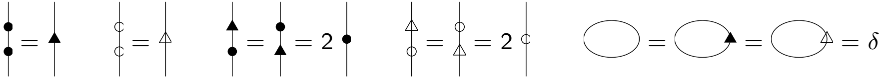


A decorated diagram algebra

Definition

We define $\widehat{\mathcal{P}}^{LR}_{n+2}(\Omega)$ to be the free $\mathbb{Z}[\delta]$-module with basis consisting of the set of LR-decorated diagrams that do not have any sequences of decorations with adjacent decorations of the same type (black or white) and do not have any of the loops listed below.

To calculate the product dd', concatenate d and d' subject to the following local relations:

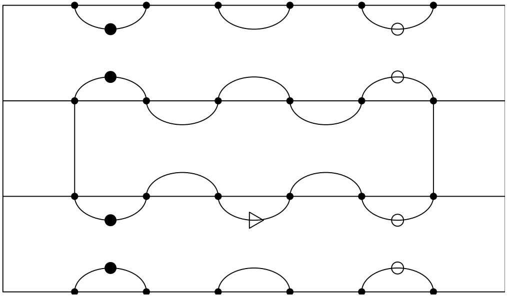


Theorem (Ernst)

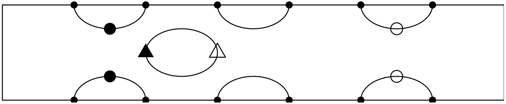
$\widehat{\mathcal{P}}^{LR}_{n+2}(\Omega)$ is a well-defined associative $\mathbb{Z}[\delta]$-algebra. A basis consists of the LR-decorated diagrams that do not have any sequences of decorations with adjacent decorations of the same type (black or white) and none of the above loops.

Example

Here is an example of diagram multiplication in $\widehat{\mathcal{P}}^{LR}_{n+2}(\Omega)$.

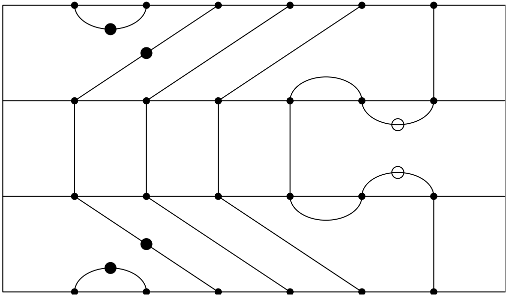


$= \quad 2$

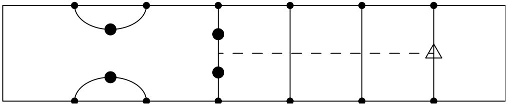


Example

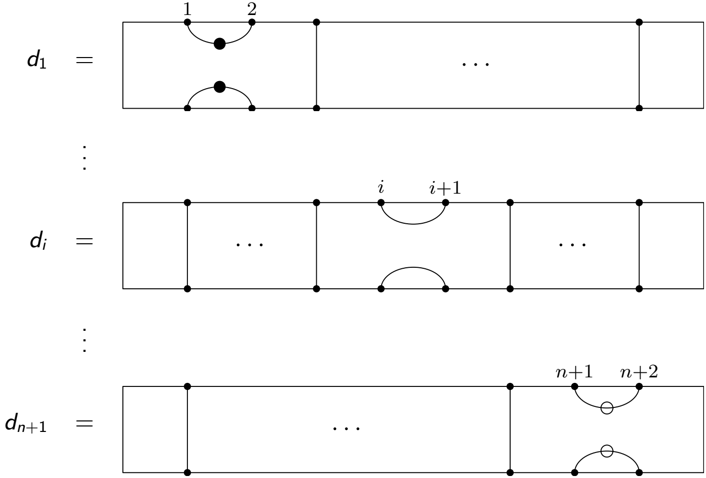
Here is another example.



$=$



Define the set of simple LR-decorated diagrams as follows:



The algebra $\widehat{\mathcal{P}}^{LR}_{n+2}(\Omega)$ is much too large to be a faithful representation of $\mathrm{TL}(\widetilde{C}_n)$.

Definition

We define $\mathbb{D}\mathrm{TL}(\widetilde{C}_n)$ to be the subalgebra of $\widehat{\mathcal{P}}^{LR}_{n+2}(\Omega)$ generated by simple diagrams.

Theorem (Ernst)

(i) *There is a description of a set of admissible diagrams that form a basis for* $\mathbb{D}TL(\widetilde{C}_n)$.

(ii) *The* $\mathbb{Z}[\delta]$*-algebra homomorphism* $\theta : \mathrm{TL}(\widetilde{C}_n) \to \mathbb{D}TL(C_n)$ *determined by*

$$\theta(b_i) = d_i$$

is an algebra isomorphism. Moreover, the admissible diagrams are in bijection with the monomial basis elements of $\mathrm{TL}(\widetilde{C}_n)$.

Proving θ is injective is the hard part! This is the first faithful diagrammatic representation of an infinite dimensional non-simply-laced (i.e., all $m(s,t) \leq 3$) generalized Temperley–Lieb algebra.