A diagrammatic representation of the Temperley–Lieb algebra

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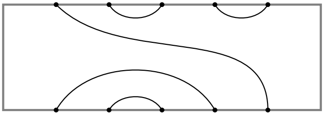
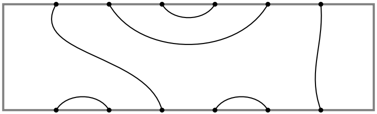
We will introduce (ordinary) Temperley–Lieb n-diagrams by way of example.

Example

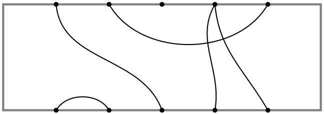
Here is an example of a 5-diagram.



Here is an example of 6-diagram.

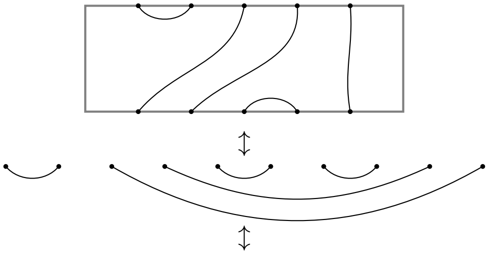


Here is an example that is not a diagram.



Fact 1

There is a one-to-one correspondence between n-diagrams and sequences of n pairs of well-formed parentheses.



()((()()))

Fact 2

It is well-known that the number of sequences of n pairs of well-formed parentheses is equal to the nth Catalan number. Therefore, the number of n-diagrams is equal to the nth Catalan number.

Comments

- The nth Catalan number is given by
$$C_n = \frac{1}{n+1}\left(\begin{array}{c}2n\\ n\end{array}\right) = \frac{(2n)!}{(n+1)!n!}.$$
- The first few Catalan numbers are $1, 1, 2, 5, 14, 42, 132$.
- Richard Stanley's book, "Enumerative Combinatorics, Vol II," contains 66 different combinatorial interpretations of the Catalan numbers. An addendum online includes additional interpretations for a grand total of 161 examples of things that are counted by the Catalan numbers.
- In this talk, we'll see one more example of where the Catalan numbers turn up.

Definition

The Temperley-Lieb algebra, TL_n, with parameter δ is the free $\mathbb{Z}[\delta]$-module having the set of n-diagrams as a basis with multiplication defined as follows.

If d and d' are n-diagrams, then dd' is obtained by identifying the "south face" of d with the "north face" of d', and then replacing any closed loops with a factor of δ.

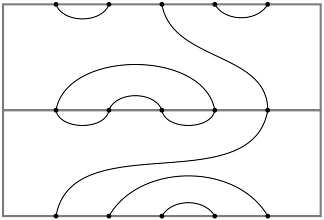
Comment

A typical element of TL_n looks like a linear combination of n-diagrams, where the coefficients in the linear combination are polynomials in δ.

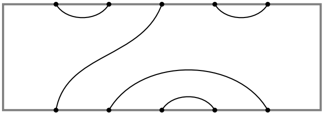
Let's look at some examples of multiplication of diagrams.

Example

Multiplication of two 5-diagrams.

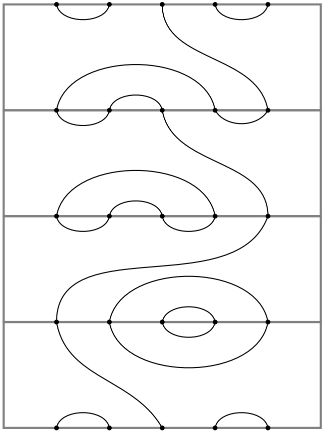


$= \delta$

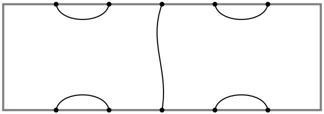


Example

And here's another example.

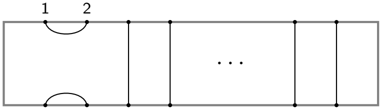


$= \delta^3$



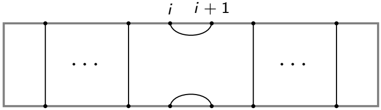
Now, we define a few "simple" n-diagrams.

d_1 =



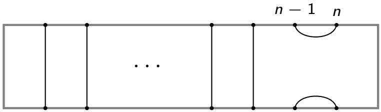
⋮

d_i =



⋮

d_{n-1} =



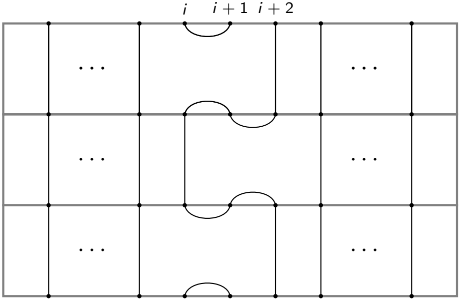
Fact

The set of "simple" diagrams generate TL_n as a unital algebra. In this case, we can write any n-diagram as a product of the "simple" n-diagrams.

Observation

We see that for $|i-j|=1$ (here $j=i+1$)

$d_i d_j d_i \quad =$



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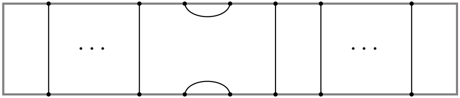


$= \quad d_i$

That is, TL_n satisfies $d_i d_j d_i = d_i$ whenever $|i-j|=1$.

Comments

- TL_n was invented in 1971 by Temperley and Lieb as an algebra with abstract generators and a presentation that includes a relation identical to the one above.
- First arose in the context of integrable Potts models in statistical mechanics.
- As well as having applications in physics, TL_n appears in the framework of knot theory, braid groups, Coxeter groups and their corresponding Hecke algebras, and subfactors of von Neumann algebras.
- Penrose/Kauffman used diagram algebra to model TL_n in 1971.
- In 1987, Vaughan Jones (awarded Fields Medal in 1990) recognized that TL_n is isomorphic to a particular quotient of the Hecke algebra of type A_{n-1} (the Coxeter group of type A_{n-1} is the symmetric group, S_n).

Now, let's consider the symmetric group, S_n. Recall that S_n is generated by the adjacent transpositions:

$$(1\ 2), (2\ 3), \ldots, (n-1\ n).$$

That is, every element of S_n can be written as a product of the adjacent transpositions.

Define

$$s_i = (i\ i+1),$$

so that $s_1, s_2, \ldots, s_{n-1}$ generate S_n.

Comments

Note that S_n satisfies the following relations:

1. $s_i^2 = 1$ for all i (transpositions are order 2)
2. $s_i s_j = s_j s_i$, for $|i-j| \geq 2$ (disjoint cycles commute)
3. $s_i s_j s_i = s_j s_i s_j$, for $|i-j| = 1$ (called a braid relation)

Notice that this last one has a similar flavor as the one we saw for TL_n, but it is different.

Definition

Let $\sigma = s_{i_1} \ldots s_{i_r} \in S_n$ be a reduced expression (i.e., we cannot do anything clever to write the expression with fewer adjacent transpositions).

We say that σ is fully commutative, or FC, if any two reduced expressions for σ may be obtained from each other by repeated commutation of adjacent generators.

Equivalently (but not immediately obvious), σ is FC iff it has no reduced expression containing $s_i s_j s_i$ for $|i - j| = 1$ (that is, there are no opportunities to apply a braid relation).

Example

The element $s_1 s_2 s_3$ is FC. However, $s_3 s_2 s_3 s_1$ is not FC because we have an opportunity to apply a braid relation.

Theorem

There is a 1-1 correspondence between the set of n-diagrams and the set of FC elements in S_n. This correspondence is induced by

$$(i\ i+1) = s_i \longmapsto d_i.$$

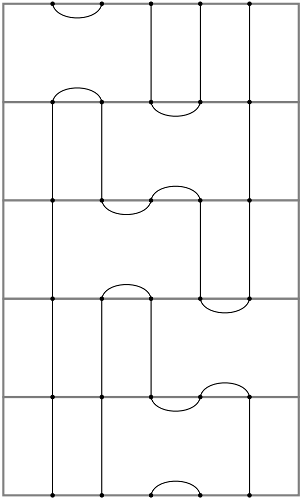
We immediately get the following corollary.

Corollary

The number of FC elements in S_n is equal to the nth Catalan number.

Example

Consider the FC element $s_1 s_3 s_2 s_4 s_3$ in S_4.



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