On an open problem of the symmetric group

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Goal

What is mathematical research? Research in mathematics takes many forms, but one common theme is that the research seeks to answer an open question concerning some collection of mathematical objects.

The goal of this talk will be to introduce you to one of the many open questions in mathematics:

> How many commutation classes does the longest element in the symmetric group have?

We will review the basics of the symmetric group and introduce all of the necessary terminology, so that we can understand this question.

Groups

Groups are fundamental objects in mathematics. Let's first remind ourselves what a group is.

Intuitive definition

Start with a static collection of objects (a set), throw in a method for combining two objects together (a binary operation) so that it satisfies some reasonable requirements (associative, identity, and inverses), and you've got yourself a group.

Whereas sets just sit there, groups have the ability for elements to interact with each other. It is this key idea that gives birth to symmetry and the simple, yet complex, beauty of group theory.

I'd say more, but I'll save it for another talk ...

The extremely vague fact

All groups "do" something. Every element of a group can be thought of as an "action."

If I want to understand a group, I think about what all of the actions are.

Combining two elements in a group together means "do the first action, then apply the second action to the result." (Yeah, just like function composition.)

OK, we need a toy to play with.

Definition

The symmetric group S_n is the collection of bijections from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$ where the operation is function composition (left $\leftarrow$ right).

What do elements of S_n "do"? Each element rearranges an arrangement of n objects. We refer to each element of S_n as a permutation.

I need 5 volunteers!

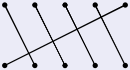
Permutation diagrams

One way of representing elements from S_n is with permutation diagrams, which are best illustrated with examples.

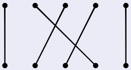
Example

Let's consider S_5. Here are some examples of permutation diagrams.

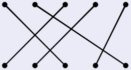
$\alpha =$



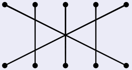
$\beta =$



$\sigma =$



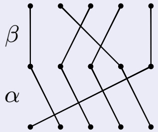
$\gamma =$

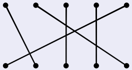


Let's try multiplying.

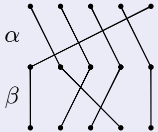
Example (continued)

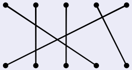
Let α and β be as on previous slide. Then

$\alpha\beta =$ 

$=$ 

But on the other hand

$\beta\alpha =$ 

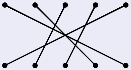
$=$ 

We see that products of permutations do not necessarily commute (order matters).

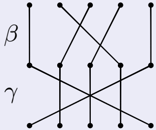
Example (continued)

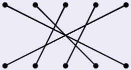
Let β and γ be as before. Then

$\beta\gamma =$ 

$=$ 

But

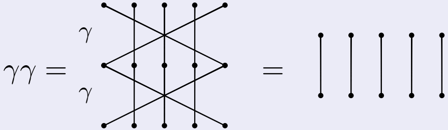
$\gamma\beta =$ 

$=$ 

Notice that these result in the same diagram. Hmmm, why did these commute?

Example (continued)

Let γ be as before. Then



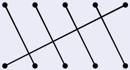
The rightmost diagram is the identity in S_5 (it's the "do nothing action"). Since $\gamma\gamma$ is equal to the identity, γ must be its own inverse.

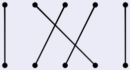
Cycle notation

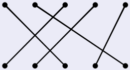
We need a more efficient way of encoding information. One way to do this is using cycle notation.

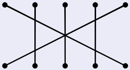
Example

Consider α, β, σ, and γ in S_5 as in the previous examples.

$\alpha =$  $= (1\ 2\ 3\ 4\ 5)$

$\beta =$  $= (2\ 4\ 3)$

$\sigma =$  $= (1\ 3)(2\ 5\ 4)$

$\gamma =$  $= (1\ 5)$

Let's try multiplying.

Example (continued)

Let α and β be as before. Then

$$\alpha\beta = (1\ 2\ 3\ 4\ 5)(2\ 4\ 3) = (1\ 2\ 5).$$

On the other hand,

$$\beta\alpha = (2\ 4\ 3)(1\ 2\ 3\ 4\ 5) = (1\ 4\ 5).$$

We could easily check that the products equal the diagrams that we found earlier. Again, we see that products of permutations do not necessarily commute (order matters).

Example (continued)

Let β and γ be as before. Then

$$\beta\gamma = (2\ 4\ 3)(1\ 5) = (1\ 5)(2\ 4\ 3) = \gamma\beta.$$

We saw earlier that β and γ commute with each other.

We've stumbled upon the following general fact.

Theorem

Products of disjoint cycles commute.

Example (continued)

Recall that $\sigma = (1\ 3)(2\ 5\ 4)$. We see that σ itself is a product of disjoint cycles. So, σ is also equal to $(2\ 5\ 4)(1\ 3)$, as well.

Let's do one last example of multiplication.

Example (continued)

As before, let $\gamma = (1\ 5)$. Then

$$\gamma\gamma = (1\ 5)(1\ 5) = (1) = \text{identity}.$$

Here's another general fact.

Theorem

2-cycles have order 2. That is, a 2-cycle is its own inverse.

The adjacent 2-cycles

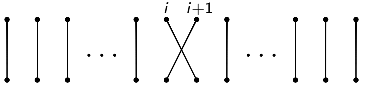
Some of the 2-cycles play a key role in what I would like to say today.

Definition

In S_n, the adjacent 2-cycles are the cycles

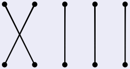
$$(1\ 2), (2\ 3), (3\ 4), \ldots, (n-2\ \ n-1), (n-1\ \ n).$$

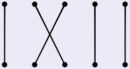
As a permutation diagram, the adjacent 2-cycle $(i\ \ i+1)$ is of the form

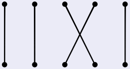
$(i\ \ i+1) =$ 

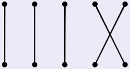
Example

The adjacent 2-cycles in S_5 are $(1\ 2), (2\ 3), (3\ 4)$, and $(4\ 5)$.

$(1\ 2) =$ 

$(2\ 3) =$ 

$(3\ 4) =$ 

$(4\ 5) =$ 

What's so special about the adjacent 2-cycles?

Theorem

Every element in S_n can be written as a product of the adjacent 2-cycles. That is, the adjacent 2-cycles generate S_n.

Unfortunately, we don't have time to discuss one of the algorithms for turning a permutation into a product of the adjacent 2-cycles.

However, it is important to note that there are potentially many different ways to express a given permutation as a product of adjacent 2-cycles.

Example

Consider the following products in S_4:

$$(1\ 2)(3\ 4)(2\ 3)(1\ 2)(2\ 3) \quad \text{and} \quad (3\ 4)(2\ 3)(1\ 2).$$

It turns out that these are both expressions for the element $(1\ 4\ 3\ 2)$. This is easily verified by just multiplying out each expression.

It will be useful for us to have methods for converting one expression into another. It turns out that we only need three "tools."

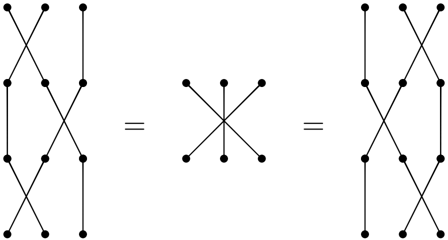
Theorem

The symmetric group S_n is generated by the adjacent 2-cycles subject only to the following relations.

1. $(i\ i+1)^2 = (1)$
   (2-cycles have order two)
2. $(i\ i+1)(j\ j+1) = (j\ j+1)(i\ i+1)$, *where* $|i-j| > 1$
   (disjoint cycles commute)
3. $(i\ i+1)(i+1\ i+2)(i\ i+1) = (i+1\ i+2)(i\ i+1)(i+1\ i+2)$
   (these are called the braid relations)

What this theorem really means is that all the symmetric group needs is the adjacent 2-cycles and these 3 rules for manipulating products of them.

We've already seen that 2-cycles have order 2 and that disjoint cycles commute. How about the braid relations?



Let's return to the previous example.

Example

We see that

$$\begin{aligned}(1\ 2)(3\ 4)(2\ 3)(1\ 2)(2\ 3) &= (1\ 2)(3\ 4)(1\ 2)(2\ 3)(1\ 2)\\ &= (1\ 2)(3\ 4)(1\ 2)(2\ 3)(1\ 2)\\ &= (3\ 4)(1\ 2)(1\ 2)(2\ 3)(1\ 2)\\ &= (3\ 4)(1\ 2)(1\ 2)(2\ 3)(1\ 2)\\ &= (3\ 4)(2\ 3)(1\ 2)\end{aligned}$$

Comments

1. If we express a permutation as a product of adjacent 2-cycles in the most efficient way possible (i.e., there is not a way to write the product with fewer factors), then we call the expression a reduced expression.
2. There may be many different reduced expressions for a given permutation, but all of them have the same number of adjacent 2-cycles occurring in the product. (This is called the length of the expression; not to be confused with cycle length.)

Example

We've already seen that

$$(1\ 2)(3\ 4)(2\ 3)(1\ 2)(2\ 3) \quad \text{and} \quad (3\ 4)(2\ 3)(1\ 2)$$

represent the same permutation. The first of these expressions is not reduced, but the second one is and has length 3.

It turns out that $(3\ 4)(2\ 3)(1\ 2)$ is the only reduced expression for $(1\ 4\ 3\ 2)$.

Theorem

Given two reduced expressions for the same permutation, we can obtain one reduced expression from the other by commuting disjoint cycles and applying braid relations.

Let's take a look at a more complicated example.

Example

The set of all reduced expressions for $(1\ 3\ 5\ 4)$ in S_5 is listed below:

$(1\ 2)(2\ 3)(1\ 2)(4\ 5)(3\ 4)$	$(1\ 2)(2\ 3)(4\ 5)(1\ 2)(3\ 4)$	$(1\ 2)(4\ 5)(2\ 3)(1\ 2)(3\ 4)$
$(1\ 2)(2\ 3)(4\ 5)(3\ 4)(1\ 2)$	$(1\ 2)(4\ 5)(2\ 3)(3\ 4)(1\ 2)$	$(4\ 5)(1\ 2)(2\ 3)(3\ 4)(1\ 2)$
$(4\ 5)(1\ 2)(2\ 3)(1\ 2)(3\ 4)$	$(2\ 3)(1\ 2)(2\ 3)(4\ 5)(3\ 4)$	$(2\ 3)(1\ 2)(4\ 5)(2\ 3)(3\ 4)$
$(2\ 3)(4\ 5)(1\ 2)(2\ 3)(3\ 4)$	$(4\ 5)(2\ 3)(1\ 2)(2\ 3)(3\ 4)$	

Commutation classes

Definition

We say that two reduced expressions are commutation equivalent if we can obtain one from the other by only commuting disjoint adjacent 2-cycles. (That is, don't apply any braid relations.)

This is an equivalence relation that partitions the set of reduced expressions for a given permutation. A single equivalence class is called a commutation class.

That is, a commutation class is a subset of reduced expressions that can be obtained from one another by commuting disjoint cycles (never apply a braid relation).

Let's return to our previous example.

Example

The set of 11 reduced expressions for (1 3 5 4) forms two commutation classes:

(1 2)(2 3)(1 2)(4 5)(3 4)	(1 2)(2 3)(4 5)(1 2)(3 4)	(1 2)(4 5)(2 3)(1 2)(3 4)
(1 2)(2 3)(4 5)(3 4)(1 2)	(1 2)(4 5)(2 3)(3 4)(1 2)	(4 5)(1 2)(2 3)(3 4)(1 2)
(4 5)(1 2)(2 3)(1 2)(3 4)		

(2 3)(1 2)(2 3)(4 5)(3 4)	(2 3)(1 2)(4 5)(2 3)(3 4)	(2 3)(4 5)(1 2)(2 3)(3 4)
(4 5)(2 3)(1 2)(2 3)(3 4)		

The longest element

We have one last thing to discuss before we can understand our open problem.

Recall that the length of a reduced expression is the number of adjacent 2-cycles appearing in the expression.

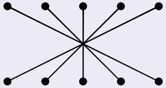
Definition

The longest element in S_n is the (unique) element having maximal length. The longest element is usually denoted by w_0.

Comments

Here are some things that are known about the longest element.

1. w_0 has length $\dfrac{n(n-1)}{2}$.
2. $w_0 = (1\ n)(2\ n-1)(3\ n-2)\cdots$ and so on.
3. The permutation diagram for w_0 is of the form ($n = 5$ here):



4. Every adjacent 2-cycle appears at least once in every reduced expression for w_0.
5. The number of reduced expressions for w_0 in S_n is known. But...

What we don't know is:

How many commutation classes does the longest element in the symmetric group have?

That is, given the set of reduced expressions for w_0, how many equivalence classes does the commutation equivalence relation partition the set into?

Of course, for a given n, we could work really hard to figure out the answer (the bigger n is, the harder we'd have to work). But what we want is a general solution.

A (good) solution would either be a function of n or a recurrence relation.

Example

In S_3, the longest element is $(1\ 3)$. In this case, there are two equivalence classes.

$$\boxed{(1\ 2)(2\ 3)(1\ 2)} \quad \boxed{(2\ 3)(1\ 2)(2\ 3)}$$

As you might expect, it gets worse pretty quick.

Example

In S_4, the longest element is $(1\ 4)(2\ 3)$. In this case, there are 8 equivalence classes.

$(2\ 3)(1\ 2)(3\ 4)(2\ 3)(3\ 4)(1\ 2)$	$(2\ 3)(3\ 4)(1\ 2)(2\ 3)(3\ 4)(1\ 2)$
$(2\ 3)(3\ 4)(1\ 2)(2\ 3)(1\ 2)(3\ 4)$	$(2\ 3)(1\ 2)(3\ 4)(2\ 3)(1\ 2)(3\ 4)$

$(1\ 2)(3\ 4)(2\ 3)(3\ 4)(1\ 2)(2\ 3)$	$(3\ 4)(1\ 2)(2\ 3)(3\ 4)(1\ 2)(2\ 3)$
$(1\ 2)(3\ 4)(2\ 3)(1\ 2)(3\ 4)(2\ 3)$	$(3\ 4)(1\ 2)(2\ 3)(1\ 2)(3\ 4)(2\ 3)$

$(1\ 2)(2\ 3)(3\ 4)(1\ 2)(2\ 3)(1\ 2)$	$(1\ 2)(2\ 3)(1\ 2)(3\ 4)(2\ 3)(1\ 2)$

$(3\ 4)(2\ 3)(3\ 4)(1\ 2)(2\ 3)(3\ 4)$	$(3\ 4)(2\ 3)(1\ 2)(3\ 4)(2\ 3)(3\ 4)$

$(2\ 3)(1\ 2)(2\ 3)(3\ 4)(2\ 3)(1\ 2)$

$(2\ 3)(3\ 4)(2\ 3)(1\ 2)(2\ 3)(3\ 4)$

$(1\ 2)(2\ 3)(3\ 4)(2\ 3)(1\ 2)(2\ 3)$

$(3\ 4)(2\ 3)(1\ 2)(2\ 3)(3\ 4)(2\ 3)$

Please send me an email if you come up with a solution for arbitrary n!

Thanks to Jason B. Hill for the very useful pdiag package, which I used to draw all of the permutation diagrams. The package can be found here

http://euclid.colorado.edu/~hilljb/pdiag/