Diagram calculus for the Temperley–Lieb algebra

Dana C. Ernst

Plymouth State University
Department of Mathematics
http://oz.plymouth.edu/~dcernst

Northeastern Section Meeting of the MAA
November 22, 2008
Definition
A *standard n-box* is a rectangle with $2n$ nodes, labeled as follows:

An *n-diagram* is a graph drawn on the nodes of a standard $n$-box such that

- Every node is connected to exactly one other node by a single edge.
- All edges must be drawn inside the $n$-box.
- The graph can be drawn so that no edges cross.
Example

Here is an example of a 5-diagram.

Here is another.
Example
Here is an example that is not a diagram.
Comment

There is a one-to-one correspondence between $n$-diagrams and sequences of $n$ pairs of well-formed parentheses.

\[
()((()()())
\]

It is well-known that the number of sequences of $n$ pairs of well-formed parentheses is equal to the $n$th **Catalan number**. Therefore, the number of $n$-diagrams is equal to the $n$th Catalan number.
The $n$th Catalan number is given by

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}.$$ 

The first few Catalan numbers are 1, 1, 2, 5, 14, 42, 132.

Richard Stanley’s book, “Enumerative Combinatorics, Vol II,” contains 66 different combinatorial interpretations of the Catalan numbers. An addendum online includes additional interpretations for a grand total of 161 examples of things that are counted by the Catalan numbers.

In this talk, we’ll see one more example of where the Catalan numbers turn up.
Definition

The Temperley-Lieb algebra, $\mathbb{TL}_n$, with parameter $\delta$ is the free $\mathbb{Z}[\delta]$-module having the set of $n$-diagrams as a basis with multiplication defined as follows.

If $d$ and $d'$ are $n$-diagrams, then $dd'$ is obtained by identifying the “south face” of $d$ with the “north face” of $d'$, and then replacing any closed loops with a factor of $\delta$.

$\mathbb{TL}_n$ is an associative algebra. That is, the multiplication of $n$-diagrams is associative.

A typical element of $\mathbb{TL}_n$ looks like a linear combination of $n$-diagrams, where the coefficients in the linear combination are polynomials in $\delta$. 
Example
Multiplication of two 5-diagrams.
Example

Here’s another example.

\[ \delta \]
Example
And here’s one more.

= $\delta^3$
Now, we define a few “simple” $n$-diagrams. Let

\[
d_1 = \begin{array}{cccccccc}
1 & 2 & 3 & 4 & \ldots & n-1 & n \\
\end{array}
\]

\[
d_i = \begin{array}{cccccccc}
1 & \ldots & i-1 & i & i+1 & i+2 & \ldots & n \\
\end{array}
\]

\[
d_{n-1} = \begin{array}{cccccccc}
1 & 2 & \ldots & n-3 & n-2 & n-1 & n \\
\end{array}
\]
Claim
The set of “simple” diagrams generate $\text{TL}_n$ as a unital algebra. In this case, we can write any $n$-diagram as a product of the “simple” $n$-diagrams.

Theorem
$\text{TL}_n$ has a presentation (as a unital algebra):
1. $d_i^2 = \delta d_i$, for all $i$
2. $d_i d_j = d_j d_i$, for $|i - j| \geq 2$
3. $d_i d_j d_i = d_i$, for $|i - j| = 1$
Let’s check that these relations actually hold.
For all \( i \), we have

\[
d_i^2 = \delta = \delta d_i
\]
For $|i - j| \geq 2$, we have

\[ d_i d_j = \]

\[ = d_j d_i \]
For $|i - j| = 1$ (here, $j = i + 1$; $j = i - 1$ being similar), we have

\[
d_i d_j d_i =
\]

\[
= d_i
\]
Comments

- $\text{TL}_n$ as an algebra with the presentation given above was invented in 1971 by Temperley and Lieb.
- First arose in the context of integrable Potts models in statistical mechanics.
- As well as having applications in physics, $\text{TL}_n$ appears in the framework of knot theory, braid groups, Coxeter groups and their corresponding Hecke algebras, and subfactors of von Neumann algebras.
- Penrose/Kauffman used diagram algebra to model $\text{TL}_n$ in 1971.
- In 1987, Vaughan Jones (awarded Fields Medal in 1990) recognized that $\text{TL}_n$ is isomorphic to a particular quotient of the Hecke algebra of type $A_{n-1}$ (the Coxeter group of type $A_{n-1}$ is the symmetric group, $S_n$).
Now, let’s consider the symmetric group, $S_n$. Recall that $S_n$ is generated by the adjacent transpositions:

$$(1\ 2), (2\ 3), \ldots, (n-1\ n).$$

That is, every element of $S_n$ can be written as a product of the adjacent transpositions.

Define

$$s_i = (i\ i+1),$$

so that $s_1, s_2, \ldots, s_{n-1}$ generate $S_n$. 
Comment
Note that $S_n$ satisfies the following relations:

1. $s_i^2 = 1$ for all $i$ (transpositions are order 2)
2. $s_i s_j = s_j s_i$, for $|i - j| \geq 2$ (disjoint cycles commute)
3. $s_i s_j s_i = s_j s_i s_j$, for $|i - j| = 1$ (called the braid relations)

In fact, we can use these relations to define $S_n$.

Also, notice that these relations look similar to the relations of $\mathcal{T}L_n$. 
Comment (continued)

Every element of $S_n$ can be written as a word in these generators and we can use the relations to potentially decrease the number of generators occurring in a word.

Example

In $S_4$

$$(1\ 2\ 3\ 4) = (1\ 2)(2\ 3)(3\ 4) = s_1s_2s_3.$$ 

This is an example of a “reduced” word in $S_4$. However, the expression

$$s_1s_3s_1s_2s_3s_1$$

is not a reduced word.

$$s_1s_3s_1s_2s_3s_1 = s_3s_1s_2s_3s_1$$
$$= s_3s_1s_1s_2s_3s_1$$
$$= s_3s_2s_3s_1$$
Example (continued)

It turns out that the last expression above is reduced. Notice that we could apply a braid relation, but this does not reduce the number of generators appearing in this expression.

\[ s_3 s_2 s_3 s_1 = s_2 s_3 s_2 s_1 \]

We can also commute \( s_1 \) and \( s_3 \), but this does not reduce the word either.

\[ s_3 s_2 s_3 s_1 = s_3 s_2 s_1 s_3 \]
Definition
Let $\sigma = s_{i_1} \ldots s_{i_r} \in S_n$ be reduced. We say that $\sigma$ is fully commutative, or FC, if any two reduced expressions for $\sigma$ may be obtained from each other by repeated commutation of adjacent generators. Equivalently (but not immediately obvious), $\sigma$ has no reduced expression containing $s_is_js_i$ for $|i - j| = 1$ (that is, there are no opportunities to apply a braid relation).

Example
In the previous example, $s_1s_2s_3$ is FC. However, $s_3s_2s_3s_1$ is not FC because we have an opportunity to apply a braid relation.
Now, consider the group algebra of the symmetric group \( S_n \) over \( \mathbb{Z} \):

\[
\mathbb{Z}[S_n]
\]

This algebra consists of linear combinations of reduced words in the generators \( s_1, \ldots, s_{n-1} \), where the coefficients in the linear combination are integers.

Next, take the two-sided ideal, \( J \), of \( \mathbb{Z}[S_n] \) generated by all elements of the form

\[
1 + s_i + s_j + s_i s_j + s_j s_i + s_i s_j s_i,
\]

where \(|i - j| = 1\) (i.e., \( s_i \) and \( s_j \) are noncommuting generators).
Now, we consider the quotient algebra $\mathbb{Z}[S_n]/J$. Let

$$b_{s_i} = (1 + s_i) + J \in \mathbb{Z}[S_n]/J.$$ 

**Definition**

If $\sigma = s_{i_1} \ldots s_{i_r}$ is reduced and FC, then

$$b_\sigma = b_{s_{i_1}} \ldots b_{s_{i_r}}$$

is a well-defined element of $\mathbb{Z}[S_n]/J$. $b_\sigma$ for $\sigma$ FC is called a monomial.
Theorem

As a unital algebra, $\mathbb{Z}[S_n]/J$ is generated by $b_{s_1}, \ldots, b_{s_{n-1}}$. Furthermore, the set $\{b_\sigma : \sigma \ FC\}$ is a free $\mathbb{Z}$-basis for $\mathbb{Z}[S_n]/J$.

That is, $\mathbb{Z}[S_n]/J$ has a basis indexed by the fully commutative elements of $S_n$. We should think of $\mathbb{Z}[S_n]/J$ as the set of all linear combinations of monomials (indexed by FC elements of $S_n$), where the coefficients of the linear combination are integers.
If we let $\delta = 2$, we have the following result.

**Theorem**

The algebras $\mathbb{Z}[S_n]/J$ and $\mathbb{T}_L n$ are isomorphic as $\mathbb{Z}$-algebras under the correspondence

$$b_{s_i} = (1 + s_i) + J \mapsto d_i.$$ 

In particular, each monomial corresponds to a unique diagram. That is, the quotient algebra $\mathbb{Z}[S_n]/J$ can be faithfully represented by the diagram algebra that we introduced earlier, where we set $\delta = 2$.

**Corollary**

Therefore, the number of FC elements in $S_n$ is equal to the $n$th Catalan number.