Proofs Without Words

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MAT 123
October 23, 2017

Warning!

Pictures can be misleading!

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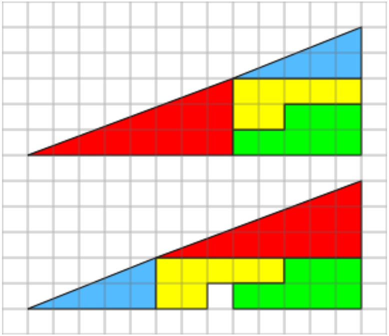

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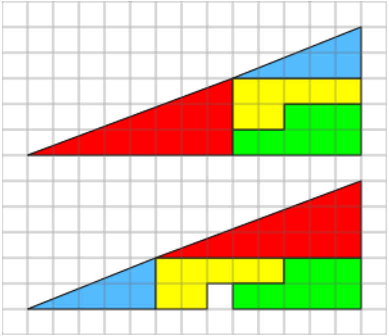


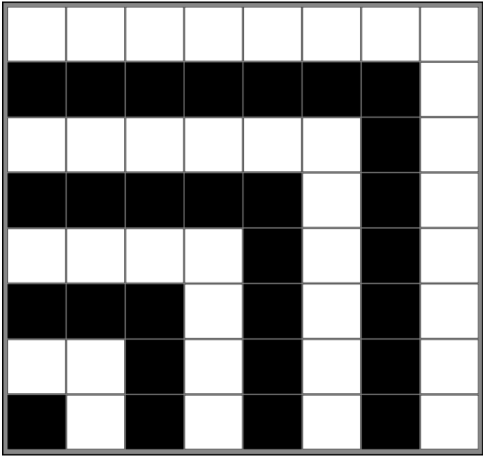
Theorem?

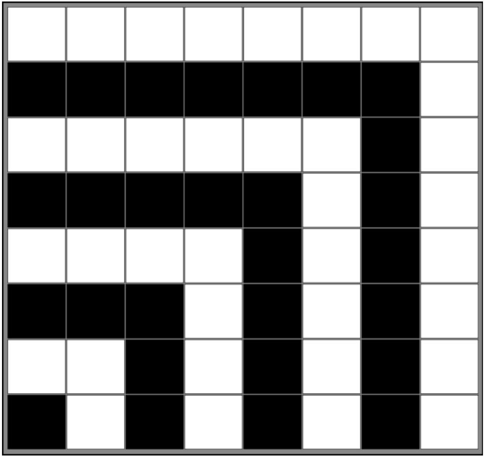
Hmmm, it looks like $32.5 = 31.5$.

Play Time

Let's play a game.

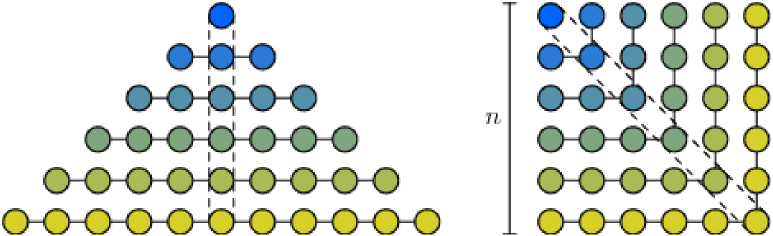
- I'll show you a picture,
- You see if you can figure out what mathematical fact it describes or proofs.

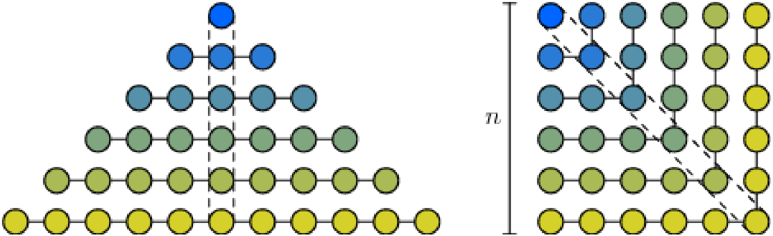




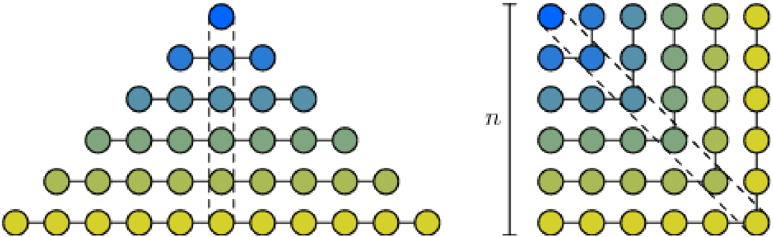
Theorem

For all $n \in \mathbb{N}$, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.


n



This the same as the previous theorem, but with a different visual proof.



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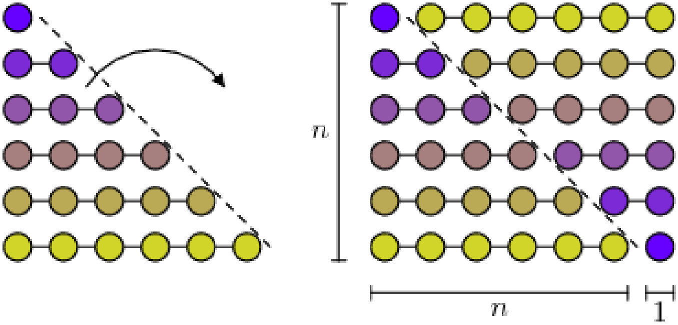
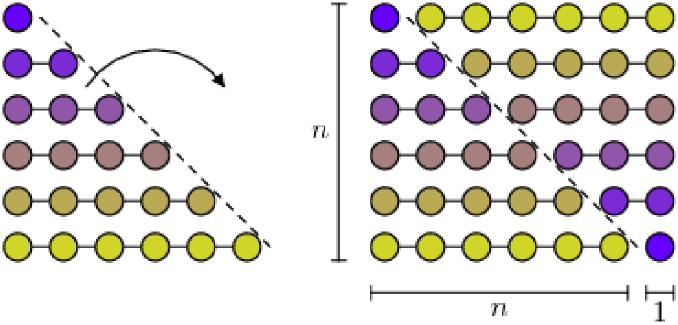
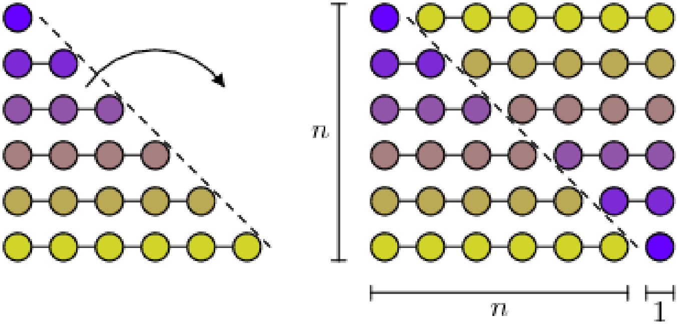


n
n
1



Theorem

For all $n \in \mathbb{N}$, $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$.

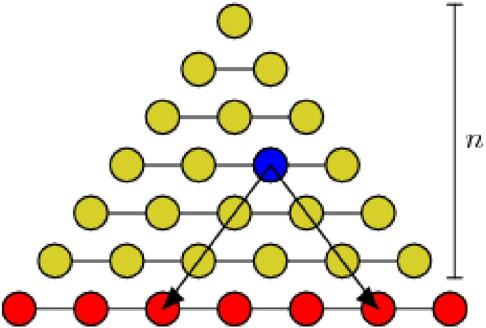


Theorem

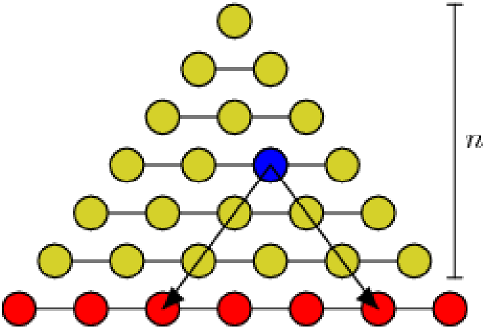
For all $n \in \mathbb{N}$, $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$.

Note

The numbers $T_n := 1 + 2 + \cdots + n$ are called **triangular numbers**.

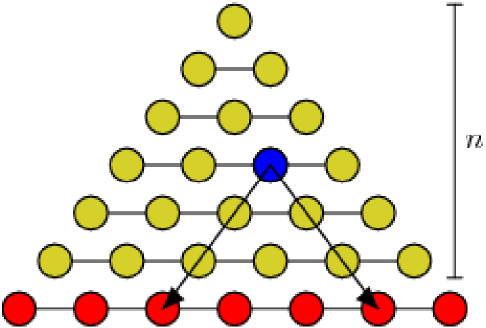


n



Theorem

For all $n \in \mathbb{N}$, $1 + 2 + \cdots + n = C(n+1, 2) := \dfrac{(n+1)!}{2!(n-1)!}$.



Theorem

For all $n \in \mathbb{N}$, $1 + 2 + \cdots + n = C(n+1, 2) := \dfrac{(n+1)!}{2!(n-1)!}$.

Corollary

For all $n \in \mathbb{N}$, $C(n+1, 2) = \dfrac{n(n+1)}{2}$.

The nth **pentagonal number** is defined to be $P_n := \frac{3n^2 - n}{2}$.

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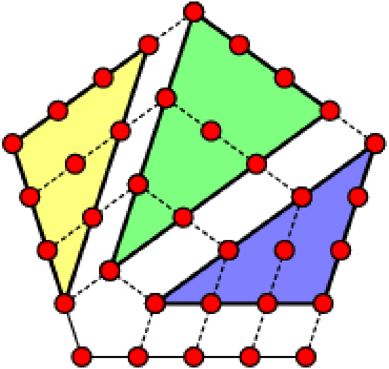

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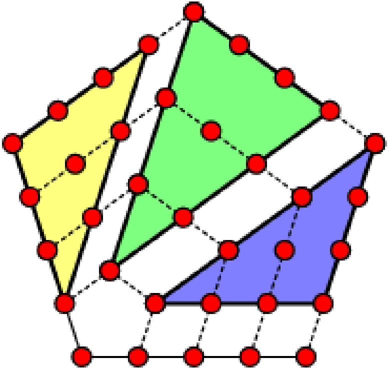
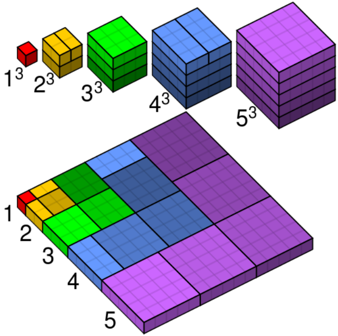
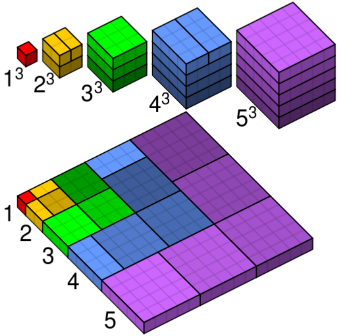


Theorem

$P_n = 3T_{n-1} + n$.

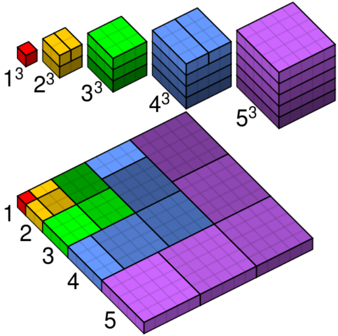


1^3
2^3
3^3
4^3
5^3
1
2
3
4
5



Theorem (Nicomachus' Theorem)

For all $n \in \mathbb{N}$, $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$.



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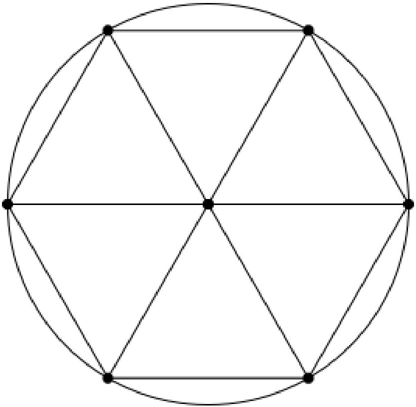
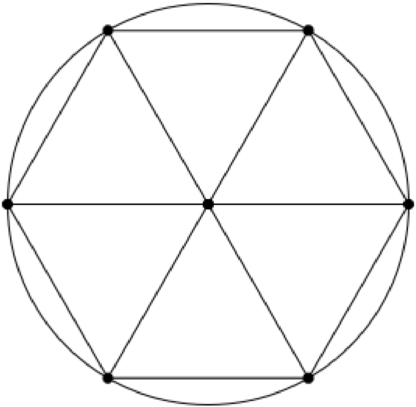
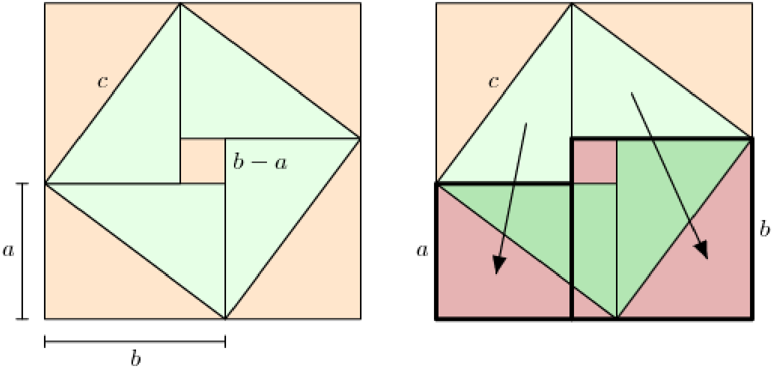
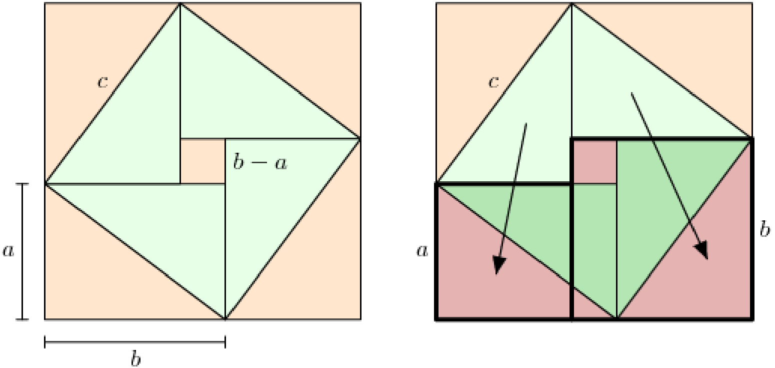




Theorem

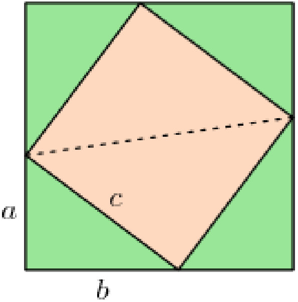
$2\pi > 6$


c
b − a
a
b
c
a
b

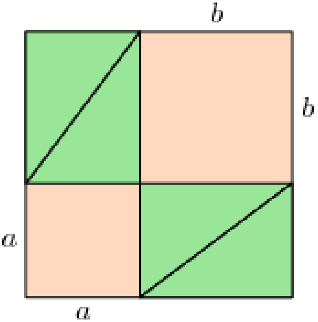


Theorem (Pythagorean Theorem)

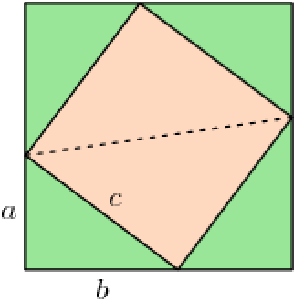
If $a, b, c \in \mathbb{N}$ are the lengths of the sides of a right triangle, where c the length of the hypotenuse, then $a^2 + b^2 = c^2$.

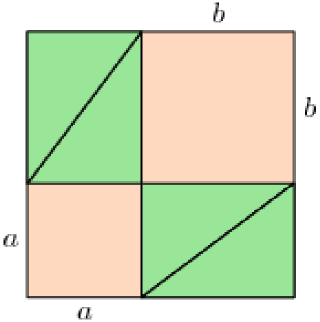


a
c
b



b
b
a
a





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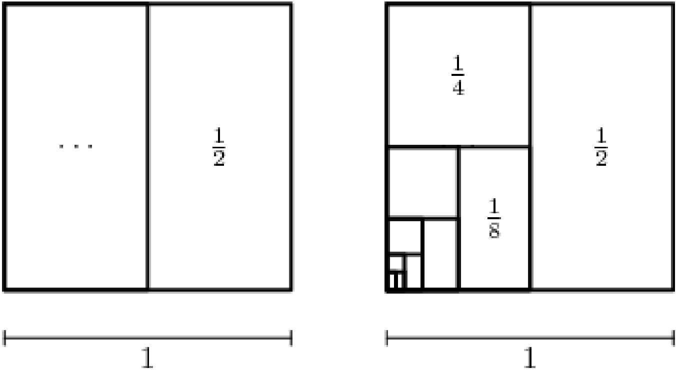
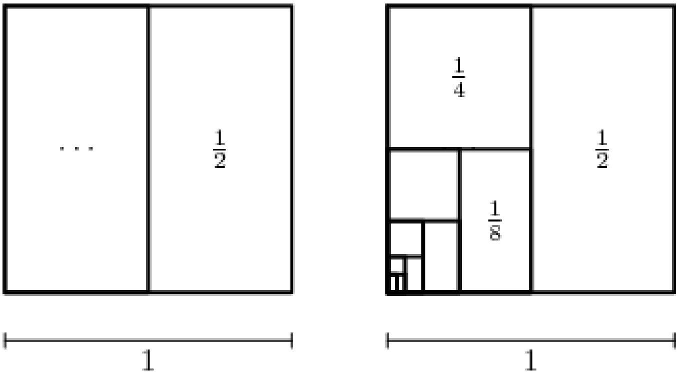


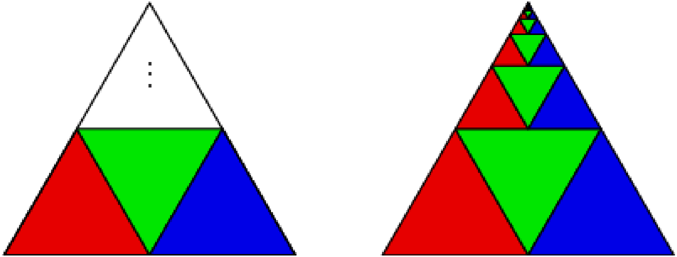
· · ·
1/2
1
1/4
1/2
1/8
1

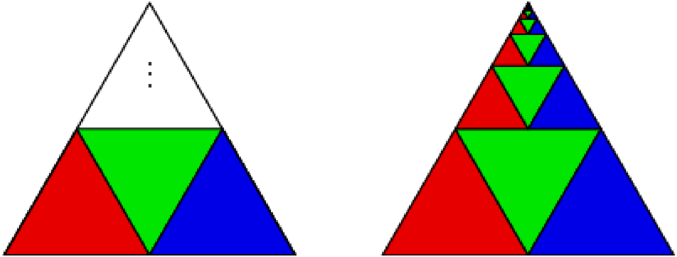


Theorem

We have the following summation formula:

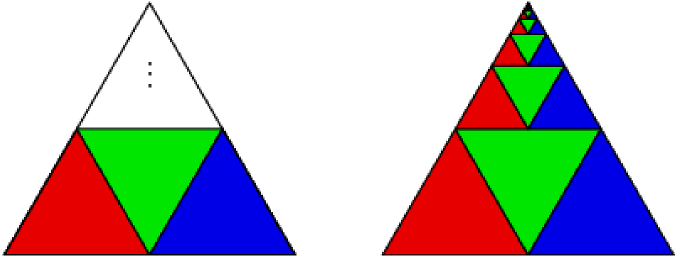
$$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k} = 1.$$





Hint

Focus on a single color.



Hint

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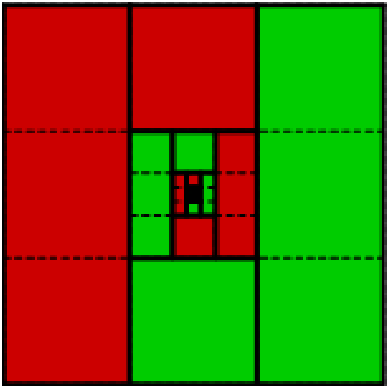
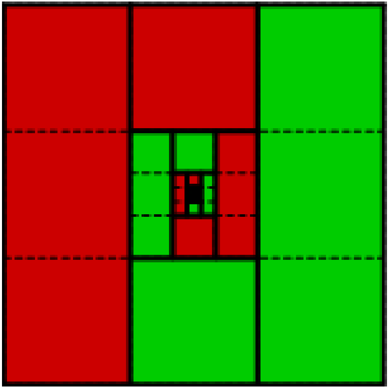
Theorem

We have the following summation formula:

$$\sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k = \frac{1}{3}.$$

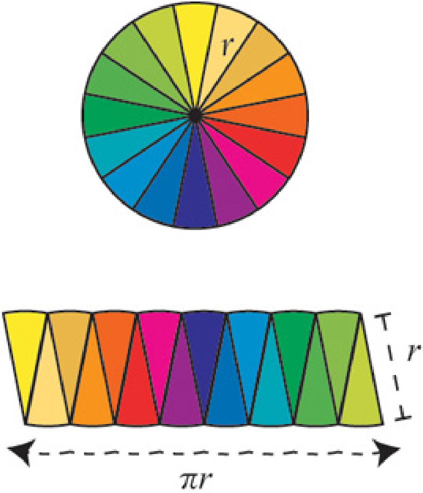


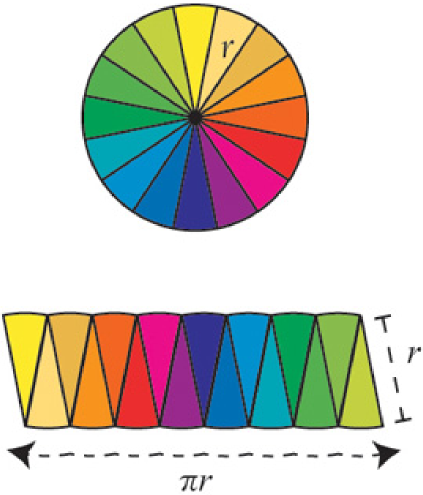


Theorem

We have the following summation formula:

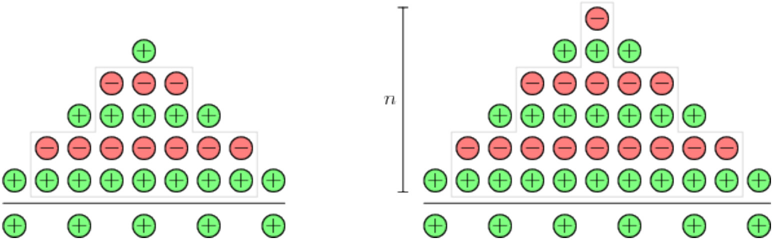
$$\sum_{k=1}^{\infty}\left(\frac{1}{3}\right)^{k}=\frac{1}{2}.$$

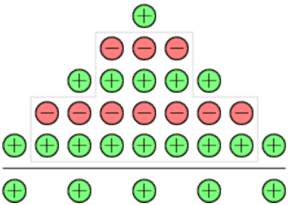

r
r
πr

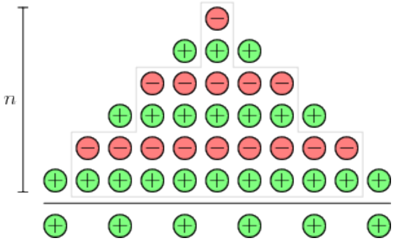


Theorem

A circle of radius r has area πr^2.


n

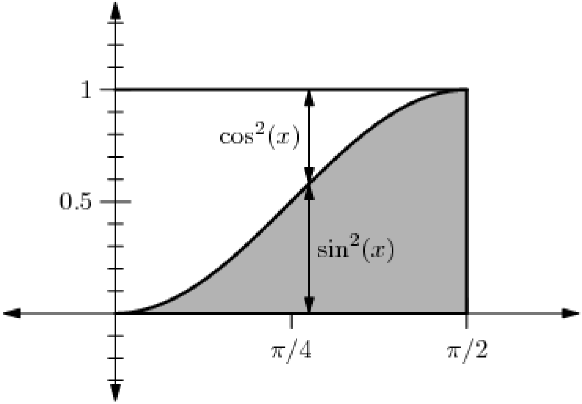


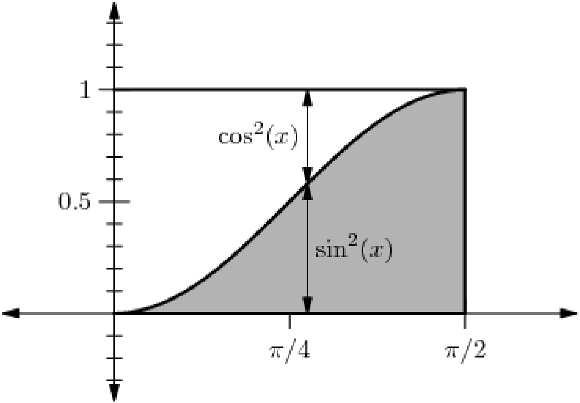


Theorem

The alternating sum of the first n odd natural numbers is n. In other words, for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} (-1)^{n-k}(2k-1) = n.$$


1
0.5
$\cos^2(x)$
$\sin^2(x)$
$\pi/4$
$\pi/2$



Theorem

We have the following fact concerning integrals:

$$\int_0^{\pi/2} \sin^2(x)\ dx = \frac{\pi}{4} = \int_0^{\pi/2} \cos^2(x)\ dx.$$

Sources

MathOverflow:
mathoverflow.net/questions/8846/proofs-without-words

Art of Problem Solving:
artofproblemsolving.com/Wiki/index.php/Proofs_without_words

Wikipedia:
en.wikipedia.org/wiki/Squared_triangular_number

Strogatz, NY Times:
opinionator.blogs.nytimes.com/2010/04/04/take-it-to-the-limit/