

Structure of braid graphs for reduced words in Coxeter groups

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Coxeter systems

Definition

A **Coxeter system** consists of a **Coxeter group** W generated by a set of involutions S together with a function $m : S \times S \rightarrow \mathbb{N} \cup \{\infty\}$ such that for $s \neq t$:

$$m(s, t) = 2 \iff st = ts \quad \left. \vphantom{m(s, t) = 2} \right\} \text{commutation relation}$$

$$\left. \begin{array}{l} m(s, t) = 3 \iff sts = tst \\ m(s, t) = 4 \iff stst = tsts \\ \vdots \end{array} \right\} \text{braid relations}$$

Reduced expressions & Matsumoto's Theorem

Definition

A word $\alpha = s_{x_1} s_{x_2} \cdots s_{x_\ell} \in S^*$ is called an **expression** for $w \in W$ if it is equal to w when considered as a group element. If ℓ is minimal among all expressions for w , α is called a **reduced expression**.

Matsumoto's Theorem

Any two reduced expressions for $w \in W$ differ by a sequence of commutation & braid moves.

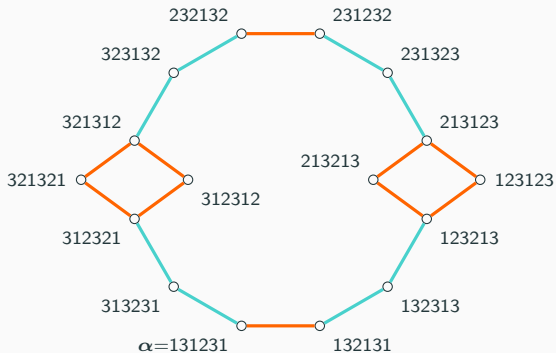
Matsumoto graphs

Example

Consider the reduced expression $\alpha = 131231$ in the Coxeter system of type D_4 .



Coxeter graph



Matsumoto graph

Braid equivalence & braid graphs

Definition

Reduced expressions α and β are **braid equivalent** iff they are related by a sequence of braid moves. The corresponding equivalence classes are called **braid classes**, denoted $[\alpha]$.

Definition

We can encode a braid class $[\alpha]$ in a **braid graph**, denoted $\mathcal{B}(\alpha)$:

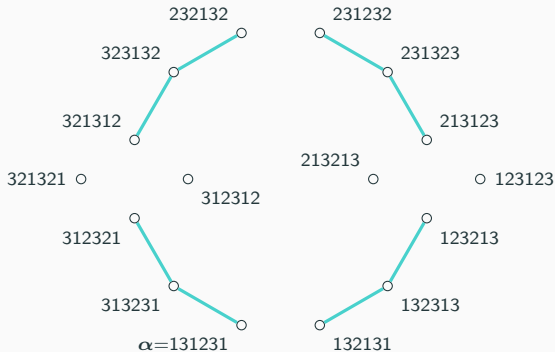
- Vertex set = $[\alpha]$
- $\{\gamma, \beta\}$ is an edge iff γ and β are related via a single **braid move**

Braid graphs are the maximal **blue** connected components in the Matsumoto graph.

Braid graphs

Example

Consider the reduced expression $\alpha = 131231$ in type D_4 from earlier.



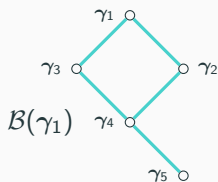
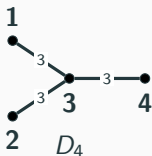
Eight braid graphs

Braid graphs (continued)

Example

In the Coxeter system of type D_4 , the expression $\gamma_1 = 2321434$ is reduced and its braid class consists of the following reduced expressions:

$$\gamma_1 = \underline{2321434}, \gamma_2 = \underline{3231434}, \gamma_3 = \underline{2321343}, \gamma_4 = \underline{323\overline{1}343}, \gamma_5 = \underline{3213143}.$$



Example of **Fibonacci cube**

Braid shadows

Notation

For $i \leq j$, we define the **interval**

$$[[i, j]] := \{i, i + 1, \dots, j - 1, j\}.$$

Definition

Let α be a reduced expression.

- $[[i, j]]$ is a **braid shadow for α** if $\alpha_{[[i, j]]} = \underbrace{st \cdots}_{m(s, t) \geq 3}$
- $\mathcal{S}(\alpha) :=$ **set of braid shadows for α**
- Collection of **braid shadows for braid class $[\alpha]$** :

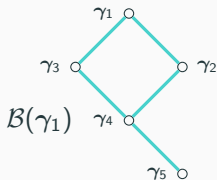
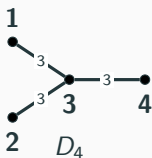
$$\mathcal{S}([\alpha]) := \bigcup_{\beta \in [\alpha]} \mathcal{S}(\beta)$$

- **rank**(α) := $|\mathcal{S}([\alpha])|$

Example

Recall the reduced expression $\gamma_1 = 2321434$ in the Coxeter system of type D_4 with braid class:

$$\gamma_1 = \underline{2321434}, \gamma_2 = \underline{3231434}, \gamma_3 = \underline{2321343}, \gamma_4 = \underline{3231\overline{343}}, \gamma_5 = \underline{3213143}.$$



We see that

$$\mathcal{S}(\gamma_1) = \{[1, 3], [5, 7]\} \text{ and } \mathcal{S}([\gamma_1]) = \{[1, 3], [3, 5], [5, 7]\}.$$

Theorem

Braid shadows are either disjoint or overlap by a single position.

Definition

If α is a reduced expression, then α is a **link** provided it either consists of a single generator or

$$\mathcal{S}([\alpha]) = \{[[1, \ell_1]], [[\ell_1, \ell_2]], \dots, [[\ell_{d-1}, \ell_d]]\}$$

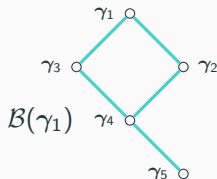
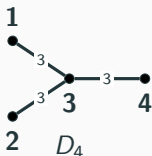
with $1 < \ell_1 < \ell_2 < \dots < \ell_d$.

Links (continued)

Example

Recall the reduced expression $\gamma_1 = 2321434$ in the Coxeter system of type D_4 with braid class:

$$\gamma_1 = \underline{2321434}, \gamma_2 = \underline{3231434}, \gamma_3 = \underline{2321343}, \gamma_4 = \underline{3231\overline{3}43}, \gamma_5 = \underline{3213143}.$$



Recall

$$\mathcal{S}([\gamma_1]) = \{[1, 3], [3, 5], [5, 7]\}.$$

So, γ_1 is a link of rank 3.

Link factorizations

Definition

If α is a reduced expression, then β is a **link factor** of α if:

- β is factor of α ,
- β is a link, and
- β is not a proper factor of a link that is a factor of α .

Theorem (ABEPV)

Every reduced expression for a nonidentity group element can be written uniquely as a product of link factors.

For emphasis, we write the **link factorization** as:

$$\alpha = \beta_1 | \beta_2 | \cdots | \beta_m.$$

Theorem (ABEPV)

If α is reduced expression with link factorization

$$\alpha = \beta_1 | \beta_2 | \cdots | \beta_m,$$

then $\mathcal{B}(\alpha)$ is the box product of the braid graphs for each β_i .

Upshot

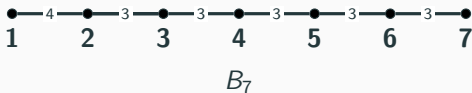
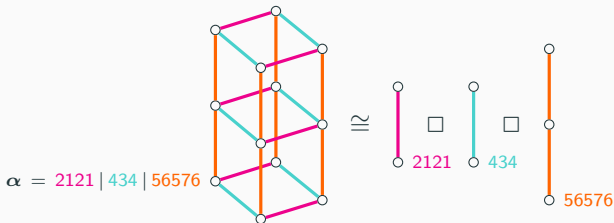
If you want to understand the structure of braid graphs, you can first characterize braid graphs for links.

Braid graphs for link factorizations (continued)

Example

Consider the reduced expression $\alpha = 212143456576$ in type B_7 with link factorization:

$$2121 \mid 434 \mid 56576.$$

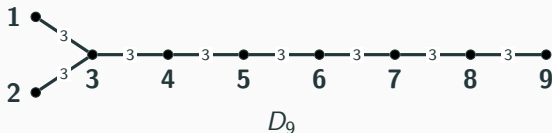
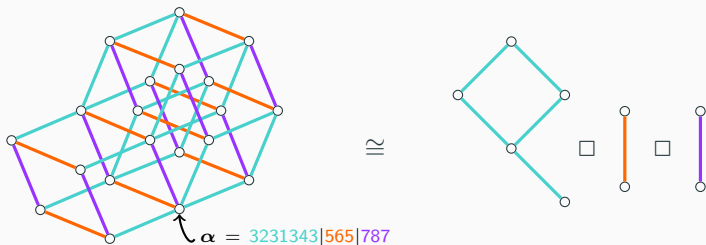


Braid graphs for link factorizations (continued)

Example

Consider the reduced expression $\alpha = 3231343565787$ in type D_9 with link factorization:

$$3231343 \mid 565 \mid 787.$$



Core of a braid shadow

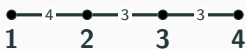
Definition

If $[[i, j]]$ is the k th braid shadow of $[\alpha]$, then the **core of the shadow at α** is the factor of α at $[[i + 1, j - 1]]$, denoted $\Theta_k(\alpha)$.

Example

Consider the reduced expression $\beta_1 = 21213243$ in the Coxeter system of type B_4 with braid class:

$$\beta_1 = \underline{2121}3243, \beta_2 = \underline{1212}\overline{32}43, \beta_3 = 1213\underline{23}\overline{43}, \beta_4 = 12132\underline{43}\overline{4}.$$



B_4

Then for example:

$$\Theta_1(\beta_1) = \underline{12}, \Theta_2(\beta_1) = \underline{3}, \Theta_3(\beta_1) = \underline{4}.$$

Definition

For $m \geq 3$, a Coxeter system (W, S) is $(3, 3, m)$ -avoiding, written Δ_m -avoiding, if its Coxeter graph avoids the following subgraph:



Theorem (ABEPV)

If (W, S) is Δ_m -avoiding and α is a link of rank at least one, then $\Theta_k(\beta)$ is an st -string or ts -string for a unique pair s and t for every $\beta \in [\alpha]$.

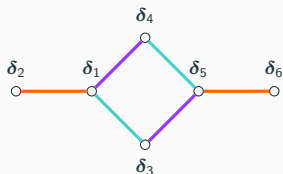
Why Δ_m -avoiding?

Example

Consider the link $\delta_1 = 12121312121$ in the Coxeter system given below with braid class:

$$\delta_1 = \underline{12121312121}, \delta_2 = 12123\overline{13}2121, \delta_3 = \underline{21212312121}$$

$$\delta_4 = \underline{12121321212}, \delta_5 = \underline{21212321212}, \delta_6 = 21213\overline{23}1212$$



Links are uniquely determined by cores

Theorem (ABEPV)

Suppose (W, S) is Δ_m -avoiding and let α and β be braid equivalent links. Then $\alpha = \beta$ iff $\Theta_k(\alpha) = \Theta_k(\beta)$ for all k .

Example

Recall the reduced expression $\beta_1 = 21213243$ in the Coxeter system of type B_4 with braid class:

$$\beta_1 = \underbrace{21213243}_{(12, 3, 4)}, \quad \beta_2 = \underbrace{121\overline{23}243}_{(21, 3, 4)}, \quad \beta_3 = \underbrace{1213\overline{23}43}_{(21, 2, 4)}, \quad \beta_4 = \underbrace{121324\overline{34}}_{(21, 2, 3)}.$$

Observation

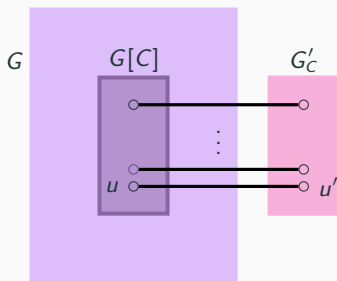
In Δ_m -avoiding Coxeter systems, $|\mathcal{B}(\alpha)| \leq 2^{\text{rank } \alpha}$.

Convex expansions

Definition

Given a graph G and a convex set $C \subseteq V(G)$, we define the **expanded graph relative to C** :

- Start with a graph G ;
- Make an isomorphic copy of $G[C]$, denoted G'_C , where each $u \in C$ corresponds to $u' \in C' := V(G'_C)$;
- For each $u \in C$, join u and u' with an edge.



Median graphs

Definition

A graph is **median** if every three vertices x, y, z have a unique median: a vertex $\text{med}(x, y, z)$ that belongs to geodesics between each pair.

Proposition (Mulder)

A graph is **median** iff it can be obtained from a single vertex by a sequence of convex expansions.

Example

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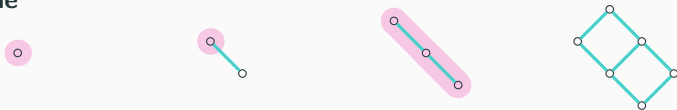
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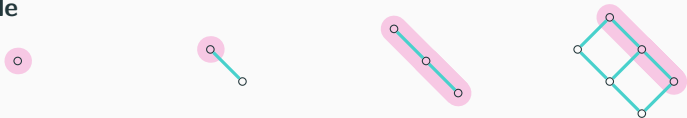
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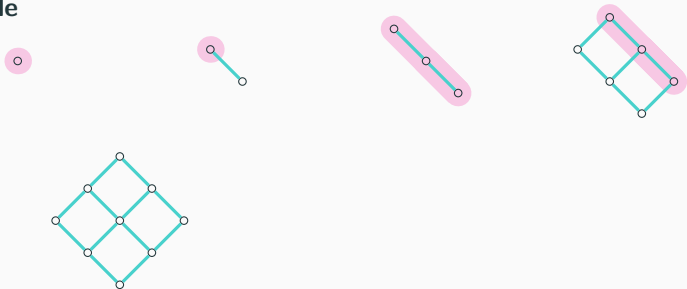
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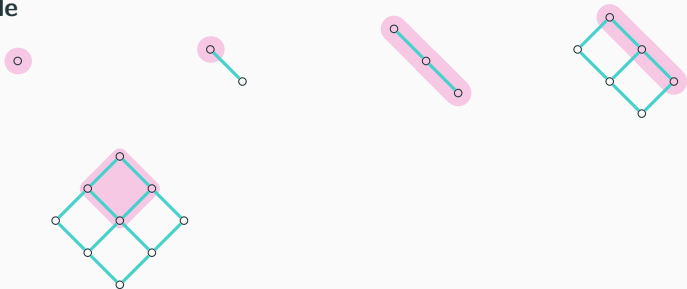
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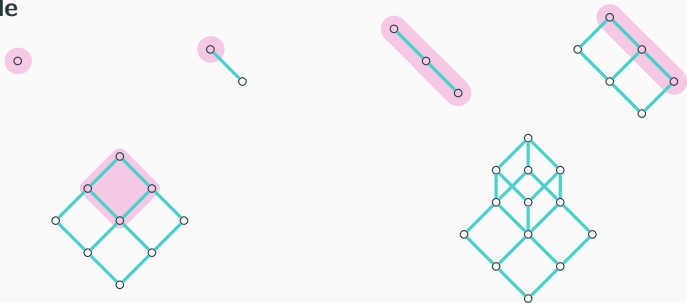
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Proposition (Mulder)

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Example



Convex expansions



Definition

Suppose (W, S) is Δ_m -avoiding and α is a link of rank $r \geq 2$, and let $\sigma \in [\alpha]$ such that the two rightmost braid shadows exist in σ . Define

$$\hat{\sigma} := \text{“chop off at last core in } \sigma \text{”}.$$

For example: $\sigma = \underline{212}\overline{3232} \Rightarrow \hat{\sigma} = 212$.

$$\text{Earth} := \{\beta \in [\alpha] \mid \Theta_r(\beta) = \Theta_r(\sigma)\}$$

$$\hat{\text{Earth}} := [\hat{\sigma}]$$

$$\text{Moon} := \{\beta \in [\alpha] \mid \Theta_r(\beta) \neq \Theta_r(\sigma)\}$$

$$\text{Shadow} := \{\beta \in \text{Earth} \mid \text{rightmost braid shadow exists in } \beta\}$$

For simplicity, we refer to the corresponding induced subgraphs using the same names.

Theorem (ABEPV)

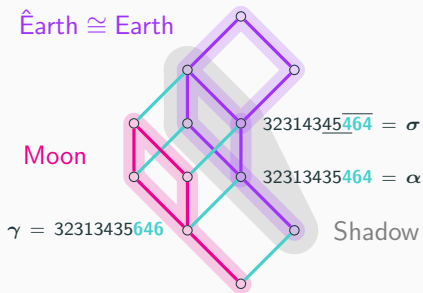
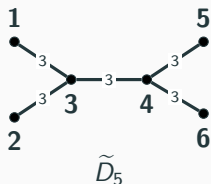
Suppose (W, S) is Δ_m -avoiding and α is a link of rank at least two. Choose $\sigma \in [\alpha]$ according to previous definition. Then

- Earth, Moon, and Shadow are convex.
- $\hat{\sigma}$ is a link with rank one less than σ .
- $\beta \in \text{Earth}$ iff $\hat{\beta} \in \hat{\text{Earth}} = [\hat{\sigma}]$.
- $\hat{\text{Earth}} \xrightarrow{\text{isometric}} \mathcal{B}(\alpha)$ with $\hat{\text{Earth}} \cong \text{Earth}$
- $\text{Shadow} \cong \text{Moon}$

Visualizing Earth, Moon, & Shadow

Example

Consider the link $\alpha = 32313435464$ in the Coxeter system of type \tilde{D}_5 .



Theorem (ABEPV)

If (W, S) is Δ_m -avoiding and α is a link, then $\mathcal{B}(\alpha)$ is median.

Outline of Proof

- We induct on rank. Base cases check out.
- Choose $\sigma \in [\alpha]$ with the last two braid shadows locally available.
- By induction, $\text{Earth} \cong \hat{\text{Earth}}$ is median.
- $\mathcal{B}(\alpha)$ is obtained from Earth via a convex expansion relative to Shadow .

Braid graphs for reduced expressions are median

Proposition

If graphs G_1 and G_2 are median, then $G_1 \square G_2$ is also median.

Theorem (ABEPV)

If (W, S) is Δ_m -avoiding and α a reduced expression, then $\mathcal{B}(\alpha)$ is median. The median of any three reduced expressions is computed by taking majority across sequence of cores.

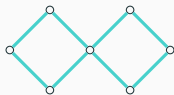
Corollary (ABEPV)

If (W, S) is Δ_m -avoiding and α a reduced expression, then $\mathcal{B}(\alpha)$ is a partial cube with isometric dimension equal to rank (also equal to diameter).

Closing remarks

Example

Not every median graph can be realized as the braid graph for a reduced expression!



Braid graphs are “special” median graphs. What is “special” ???

Hot off the press!

If (W, S) is Δ_m -avoiding and α is a reduced expression, then $\mathcal{B}(\alpha)$ is the underlying graph for the Hasse diagram of a distributive lattice. But not every distributive lattice arises in this way!

To do!

Deal with the pesky Δ_m -avoiding obstruction! We conjecture that braid graphs are still median (and distributive lattice?) in this context.