Diagram calculus for the Temperley–Lieb algebra

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Definition

A *standard n-box* is a rectangle with $2n$ nodes, labeled as follows:

- Every node is connected to exactly one other node by a single edge.
- All edges must be drawn inside the $n$-box.
- The graph can be drawn so that no edges cross.

An *$n$-diagram* is a graph drawn on the nodes of a standard $n$-box such that

- Every node is connected to exactly one other node by a single edge.
- All edges must be drawn inside the $n$-box.
- The graph can be drawn so that no edges cross.
Example
Here is an example of a 5-diagram.

Here is another.
Example
Here is an example that is not a diagram.
Comment
There is a one-to-one correspondence between $n$-diagrams and sequences of $n$ pairs of well-formed parentheses.

It is well-known that the number of sequences of $n$ pairs of well-formed parentheses is equal to the $n$th Catalan number. Therefore, the number of $n$-diagrams is equal to the $n$th Catalan number.
Comment (continued)

- The \textit{n}th Catalan number is given by

\[
C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}.
\]

- The first few Catalan numbers are 1, 1, 2, 5, 14, 42, 132.

- Richard Stanley’s book, “Enumerative Combinatorics, Vol II,” contains 66 different combinatorial interpretations of the Catalan numbers. An addendum online includes additional interpretations for a grand total of 161 examples of things that are counted by the Catalan numbers.

- In this talk, we’ll see one more example of where the Catalan numbers turn up.
Definition

The Temperley-Lieb algebra, \( \mathbb{T}L_n(\delta) \), with parameter \( \delta \) is the free \( \mathbb{Z}[\delta] \)-module having the set of \( n \)-diagrams as a basis with multiplication defined as follows.

If \( d \) and \( d' \) are \( n \)-diagrams, then \( dd' \) is obtained by identifying the “south face” of \( d \) with the “north face” of \( d' \), and then replacing any closed loops with a factor of \( \delta \).

\( \mathbb{T}L_n \) is an associative algebra. That is, the multiplication of \( n \)-diagrams is associative.
Comment

- $\mathbb{Z}[\delta]$ is the set of all polynomials in $\delta$ with integer coefficients. For example,
  \[
  \delta^3 - 4\delta + 1 \in \mathbb{Z}[\delta].
  \]

- In this context, we should think of an algebra as being like a vector space, except we can also multiply the “vectors,” which in this case are diagrams. Also, everything here is happening over $\mathbb{Z}[\delta]$ instead of a field.

- A typical element of $\text{TL}_n(\delta)$ looks like a linear combination of $n$-diagrams, where the coefficients in the linear combination are polynomials in $\delta$.

- Let’s look at some examples of multiplication of diagrams.
Example

Multiplication of two 5-diagrams.
Example
Here’s another example.

\[ \delta \]
Example
And here’s one more.

= \delta^3
Theorem

In general, the product of any number of $n$-diagrams will be equal to

\[ \delta^k \text{ some } n\text{-diagram} \]

where $0 \leq k < \infty$. Note that $k = 0$ if there are no loops in the product.

Now, we define a few “simple” $n$-diagrams. These diagrams will form a generating set for $\mathcal{TL}_n(\delta)$. 
Let

\[ d_1 = \]

\[ \vdots \]

\[ d_i = \]

\[ \vdots \]

\[ d_{n-1} = \]
Claim
The set of “simple” diagrams generate $\text{TL}_n(\delta)$ as a unital algebra. In this case, we can write any $n$-diagram as a product of the “simple” $n$-diagrams.

Theorem
$\text{TL}_n(\delta)$ has a presentation (as a unital algebra):

1. $d_i^2 = \delta d_i$, for all $i$
2. $d_i d_j = d_j d_i$, for $|i - j| \geq 2$
3. $d_i d_j d_i = d_i$, for $|i - j| = 1$

Let’s check that these relations actually hold.
For all $i$, we have

$$d_i^2 = \delta = \delta d_i$$
For $|i - j| \geq 2$, we have

\[
d_i d_j = \quad = d_j d_i
\]
For $|i - j| = 1$ (here, $j = i + 1$; $j = i - 1$ being similar), we have

\[ d_i d_j d_i = d_i \]
Comments

- \( \mathrm{TL}_n(\delta) \) as an algebra with the presentation given above was invented in 1971 by Temperley and Lieb.
- First arose in the context of integrable Potts models in statistical mechanics.
- As well as having applications in physics, \( \mathrm{TL}_n(\delta) \) appears in the framework of knot theory, braid groups, Coxeter groups and their corresponding Hecke algebras, and subfactors of von Neumann algebras.
- Penrose/Kauffman used diagram algebra to model \( \mathrm{TL}_n(\delta) \) in 1971.
- In 1987, Vaughan Jones (awarded Fields Medal in 1990) recognized that \( \mathrm{TL}_n(\delta) \) is isomorphic to a particular quotient of the Hecke algebra of type \( A_{n-1} \) (the Coxeter group of type \( A_{n-1} \) is the symmetric group, \( S_n \)).
Example

$\text{TL}_3(\delta)$ is generated by $d_1$ and $d_2$, where these generators satisfy the relations

\begin{align*}
   d_1^2 &= \delta d_1 \text{ and } d_2^2 = \delta d_2 \\
   d_1 d_2 d_1 &= d_1 \text{ and } d_2 d_1 d_2 = d_2
\end{align*}

Example

$\text{TL}_4(\delta)$ is generated by $d_1, d_2$, and $d_3$ where these generators satisfy the relations

\begin{align*}
   d_1^2 &= \delta d_1, d_2^2 = \delta d_2, \text{ and } d_3^2 = \delta d_3 \\
   d_1 d_3 &= d_3 d_1 \\
   d_1 d_2 d_1 &= d_1 \text{ and } d_2 d_1 d_2 = d_2 \\
   d_2 d_3 d_2 &= d_2 \text{ and } d_3 d_2 d_3 = d_2
\end{align*}
Theorem
A basis for $\text{TL}_n$ may be described in terms of “reduced words” in the algebra generators $d_i$.

Example
Consider the following expression in $\text{TL}_4(\delta)$.

$$d_1 d_3 d_1 d_2 d_3.$$  

This expression is not “reduced”.
Example (continued)

\[ d_1 d_3 d_1 d_2 d_3 = d_3 d_1 d_1 d_2 d_3 \]
\[ = d_3 d_1 d_1 d_2 d_3 \]
\[ = \delta d_3 d_1 d_2 d_3 \]
\[ = \delta d_3 d_1 d_2 d_3 \]
\[ = \delta d_1 d_3 d_2 d_3 \]
\[ = \delta d_1 d_3 d_2 d_3 \]
\[ = \delta d_1 d_3 \]

The expression \( d_1 d_3 \) is “reduced” and represents a basis element of \( TL_3(\delta) \). Note that it’s not the only reduced expression for this basis element.

\[ d_1 d_3 = d_3 d_1 \]
The symmetric group $S_n$

Now, let’s consider the symmetric group, $S_n$. Recall that $S_n$ is generated by the adjacent transpositions:

$$(1\ 2), (2\ 3), \ldots, (n - 1\ n).$$

That is, every element of $S_n$ can be written as a product of the adjacent transpositions.

Now, define

$$s_i = (i\ i + 1).$$

**Example**

$S_4$ is generated by

$$s_1 = (1\ 2), s_2 = (2\ 3), s_3 = (3\ 4).$$
Comment

Note that $S_n$ satisfies the following relations:

1. $s_i^2 = 1$ for all $i$ (transpositions are order 2)
2. $s_is_j = s_js_i$, for $|i - j| \geq 2$ (disjoint cycles commute)
3. $s_is_js_i = s_js_is_j$, for $|i - j| = 1$ (called the braid relations)

In fact, we can use these relations to define $S_n$. Also, notice that these relations look similar to the relations of $\mathcal{TL}_n(\delta)$. 
Comment (continued)

Every element of $S_n$ can be written as a word in these generators and we can use the relations to potentially decrease the number of generators occurring in a word.

Example

In $S_4$

$$(1 \ 2 \ 3 \ 4) = (1 \ 2)(2 \ 3)(3 \ 4) = s_1 s_2 s_3.$$

This is an example of a “reduced” word in $S_4$. However, the expression

$$s_1 s_3 s_1 s_2 s_3 s_1$$

is not a reduced word.

$$s_1 s_3 s_1 s_2 s_3 s_1 = s_3 s_1 s_2 s_3 s_1$$

$$= s_3 s_1 s_2 s_3 s_1$$

$$= s_3 s_2 s_3 s_1$$
Example (continued)
The last expression above is reduced. Notice that we could apply a braid relation in the last expression above, but it does not reduce the last expression above.

\[ s_3 s_2 s_3 s_1 = s_2 s_3 s_2 s_1 \]

We can also commute \( s_1 \) and \( s_3 \), but this does not reduce the word either.

\[ s_3 s_2 s_3 s_1 = s_3 s_2 s_1 s_3 \]
Definition
Let $\sigma = s_{i_1} \ldots s_{i_r} \in S_n$ be reduced. We say that $\sigma$ is fully commutative, or FC, if any two reduced expressions for $\sigma$ may be obtained from each other by repeated commutation of adjacent generators. In other words, $\sigma$ has no reduced expression containing $s_is_js_i$ for $|i - j| = 1$ (that is, there are no opportunities to apply a braid relation).

Example
In the previous example, $s_1s_2s_3$ is FC. However, $s_3s_2s_3s_1$ is not FC because we have an opportunity to apply a braid relation.
Now, consider the group algebra of the symmetric group $S_n$ over $\mathbb{Z}$:

$$\mathbb{Z}[S_n]$$

This algebra consists of linear combinations of reduced words in the generators $s_1, \ldots, s_{n-1}$, where the coefficients in the linear combination are integers. For example,

$$s_1 s_2 + 3s_2 s_3 s_2 \in \mathbb{Z}[S_4].$$

**Comment**

The elements of $S_n$ form a free $\mathbb{Z}$-basis for $\mathbb{Z}[S_n]$. 
Next, take the two-sided ideal, $J$, of $\mathbb{Z}[S_n]$ generated by all elements of the form

$$1 + s_i + s_j + s_is_j + s_js_i + s_is_js_i,$$

where $|i - j| = 1$ (i.e., $s_i$ and $s_j$ are noncommuting generators).

**Example**
Consider $\mathbb{Z}[S_3]$. In this case, $J$ is generated by

$$1 + s_1 + s_2 + s_1s_2 + s_2s_1 + s_1s_2s_1.$$

What this means is that $J$ is the smallest ideal containing the linear combination above (it is closed under multiplication on the left and right by $\mathbb{Z}$-linear combinations of elements from $S_n$).
Now, we consider the quotient algebra $\mathbb{Z}[S_n]/J$. Let

$$b_{s_i} = (1 + s_i) + J \in \mathbb{Z}[S_n]/J.$$ 

**Definition**

If $\sigma = s_{i_1} \ldots s_{i_r}$ is reduced and FC, then

$$b_\sigma = b_{s_{i_1}} \ldots b_{s_{i_r}}$$

is a well-defined element of $\mathbb{Z}[S_n]/J$. $b_\sigma$ for $\sigma$ FC is called a monomial.
Theorem

As a unital algebra, $\mathbb{Z}[S_n]/J$ is generated by $b_{s_1}, \ldots, b_{s_{n-1}}$. Furthermore, the set $\{b_{\sigma} : \sigma \text{ FC}\}$ is a free $\mathbb{Z}$-basis for $\mathbb{Z}[S_n]/J$.

That is, $\mathbb{Z}[S_n]/J$ has a basis indexed by the fully commutative elements of $S_n$. We should think of $\mathbb{Z}[S_n]/J$ as the set of all linear combinations of monomials (indexed by FC elements of $S_n$), where the coefficients of the linear combination are integers.
If we let $\delta = 2$, we have the following result.

**Theorem**

The algebras $\mathbb{Z}[S_n]/J$ and $T\mathcal{L}_n(2)$ are isomorphic as $\mathbb{Z}$-algebras under the correspondence

$$b_{s_i} = (1 + s_i) + J \mapsto d_i.$$  

That is, the quotient algebra $\mathbb{Z}[S_n]/J$ can be represented by the diagram algebra that we introduced earlier, where we set $\delta = 2$.

**Corollary**

Therefore, the number of FC elements in $S_n$ is equal to the $n$th Catalan number.