My Sabbatical Report

NAU Mathematics & Statistics Colloquium

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What did I do on sabbatical?

- Combinatorial Game Theory:
    - Paper with Benesh and Sieben revised, accepted, and published.
    - Paper with Benesh, Meyer, Salmon, Sieben submitted, and a second paper in progress.
    - Attendance and talk at *Combinatorial Game Theory Colloquium* in Azores, Portugal. Initiated project with Bašić, Ellis, Popović, Sieben.
    - Talk in *Virtual CGT Seminar*.
    - Two talks in ACGT.
- Enumerative Combinatorics:
    - Spent most of June in Iceland collaborating with Anders Claesson and Giulio Cerbai at the University of Iceland. Project(s) underway.
    - Gave talk at the University of Iceland.

What did I do on sabbatical? (continued)

- SoTL and Professional Development:
  - Paper with J. Slye about teaching abstract algebra submitted.
  - Co-facilitator for OPEN Math professional development project.
  - Paper about OPEN Math project with Laursen, Hitchman, Jones, Yoshinobu submitted (and accepted pending revisions).
- Open Educational Resources:
  - Finalized Instructor Manual for *An Introduction to Proof via Inquiry-Based Learning*.
  - Applied for and awarded NAU OER mini-grant to convert book(s) from LaTeX to PreTeXt. Wrapped up this work for one book a few weeks ago.
  - Attended *Getting Started with PreTeXt Virtual Workshop*.
  - Attended *2023-2024 Institute on Open Educational Resources Kick-Off Event*.

What didn't I do?

- I am NOT rejuvenated!
- I have a paper concerning the combinatorics of genome rearrangements mostly written that I barely touched. This is the one project I was most excited about wrapping up...
- Falk and I have a project we have been tinkering with for ages. I had hoped that maybe I would have time to commit to this near the end of my sabbatical. Didn't happen.

Let's look at some pictures

Categories of rulesets and games

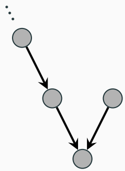
Impartial rulesets and games

- We model games with special digraphs. We call the vertices of a digraph positions and the set $\mathrm{Opt}(p)$ of out-neighbors of a position p the options of p.
- An impartial ruleset R is a digraph with no infinite directed walks.
- A position p is called terminal if $\mathrm{Opt}(p) = \emptyset$. We say that q is a subposition of p if there is a directed walk from p to q.
- An impartial game G is an impartial ruleset that has a unique source vertex called the starting position and every position is a subposition of the starting position.

Impartial rulesets and games (continued)

Example

A ruleset with a unique source that is not a game.

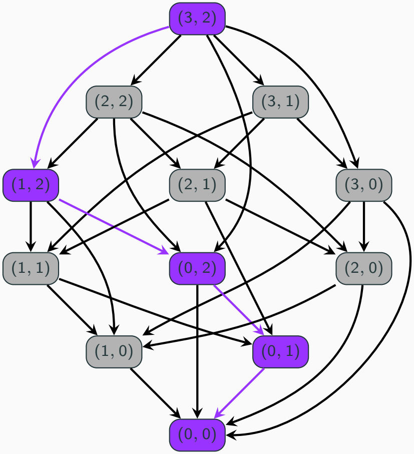


Impartial rulesets and games (continued)

- In a play of a game G, two players take turns replacing the current position with one of its options. At beginning of play: current position is starting position. Game ends when current position becomes a terminal position.
- To determine the winner at end of a play we need a winning condition, which amounts to classifying each terminal position as winning or losing.
- Common choices:
  - Normal play: player unable to move loses.
  - Misere play: player unable to move wins.
- Choice of a winning condition determines an outcome function o on the positions of a ruleset R.
- The outcome of a game G is $o(\mathrm{G}) := o(s)$, where s is the starting position.

Example

Two-pile NIM with a play highlighted in purple.



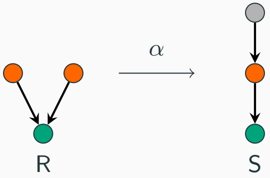
G

Option preserving maps

- A digraph map $\alpha : C \to D$ between digraphs C and D is a function $\alpha : V(C) \to V(D)$. If α is is a map between rulesets or games, then we say α is a ruleset map or game map, respectively.
- A digaph map $\alpha : C \to D$ is option preserving if it satisfies $\mathrm{Opt}(\alpha(p)) = \alpha(\mathrm{Opt}(p))$.
- A game map $\alpha : \mathsf{G} \to \mathsf{H}$ is source preserving if it takes the starting position of G to the starting position of H.

Example

An option preserving ruleset map $\alpha : \mathsf{R} \to \mathsf{S}$.

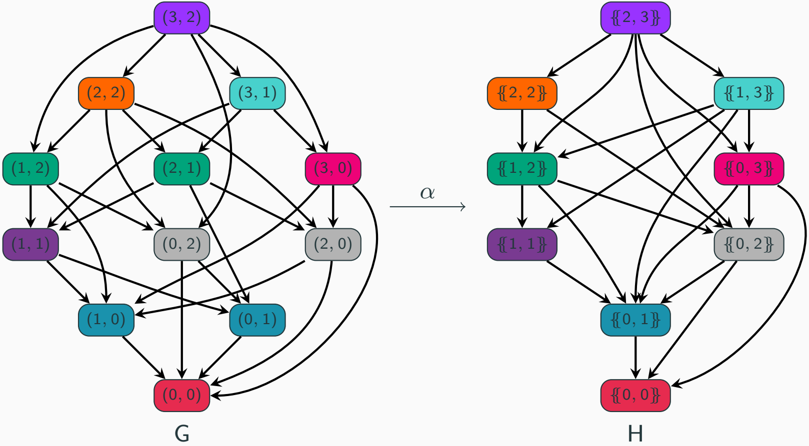


Note that S is actually a game.

Option preserving maps (continued)

Example

The game of NIM played on several piles can be modeled using ordered pairs or multisets. The mapping $\alpha : \mathsf{G} \to \mathsf{H}$ defined via $\alpha(a_1, \ldots, a_n) = \{\!\{a_1, \ldots, a_n\}\!\}$ between these two approaches is both option and source preserving.



Option preserving maps (continued)

Proposition

If $\alpha : \mathsf{R} \to \mathsf{S}$ is an option preserving ruleset map, then α takes arrows to arrows and every arrow connecting positions in $\alpha(V(\mathsf{R}))$ is the image of an arrow.

Proposition

If $\alpha : \mathsf{G} \to \mathsf{H}$ is a source and option preserving game map, then $\mathrm{nim}(\mathsf{G}) = \mathrm{nim}(\mathsf{H})$. Outcome and formal birthday (i.e., "height" of a position) are also preserved.

Proposition

If $\alpha : \mathsf{R} \to D$ is an option preserving map from a ruleset to a digraph, then the digraph induced by $\alpha(V(\mathsf{R}))$ in D is a ruleset. Moreover if R is a game, then $\alpha(V(\mathsf{R}))$ is also a game.

Quotient rulesets

An equivalence relation $\sim$ on the positions of a ruleset R is a congruence relation if $p \sim q$ implies $[\mathrm{Opt}(p)] = [\mathrm{Opt}(q)]$; with the interpretation $[\mathrm{Opt}(x)] = \{[y] \mid y \in \mathrm{Opt}(x)\}$.

Proposition

If $\alpha : \mathsf{R} \to \mathsf{S}$ is an option preserving ruleset map, then the kernel of α is a congruence relation.

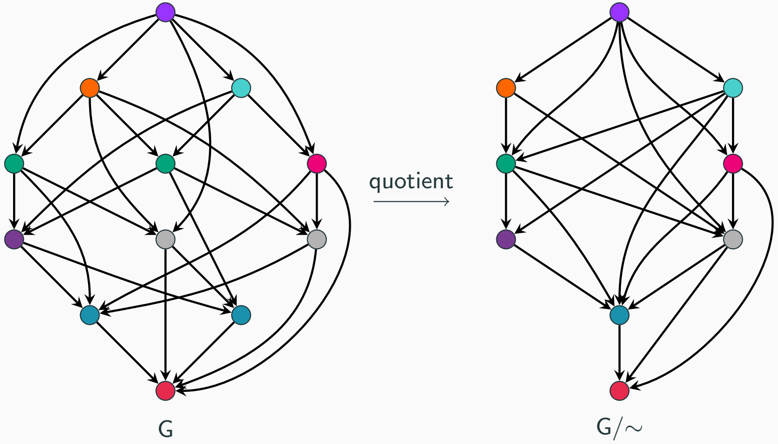
Let $\sim$ be a congruence relation on the positions of ruleset R. The quotient digraph $\mathsf{R}/\sim$ has vertex set $V(\mathsf{R})/\sim$ and arrow set $\{([p], [q]) \mid q \in \mathrm{Opt}(p)\}$.

Proposition

If R is a ruleset and $\sim$ a congruence relation, then $\mathsf{R}/\sim$ is a ruleset. In particular, if R is a game with starting position s, then $\mathsf{R}/\sim$ is a game whose starting position is $[s]$.

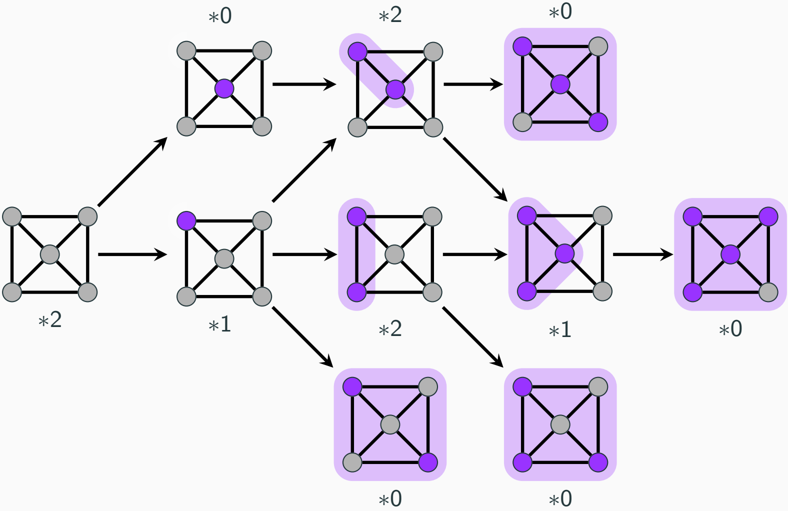
We refer to $\mathsf{R}/\sim$ as a quotient ruleset, respectively quotient game.

Example



Example

Quotient determined by symmetry:



First Isomorphism Theorem

The following result can be thought of as the First Isomorphism Theorem for rulesets/games.

Proposition

- If $\alpha : \mathsf{R} \to \mathsf{S}$ is an option preserving ruleset map, then $\alpha(\mathsf{R})$ is isomorphic to the quotient ruleset R/α.
- If $\alpha : \mathsf{G} \to \mathsf{H}$ is a source and option preserving game map, then G/α is isomorphic to H.

Slim rulesets

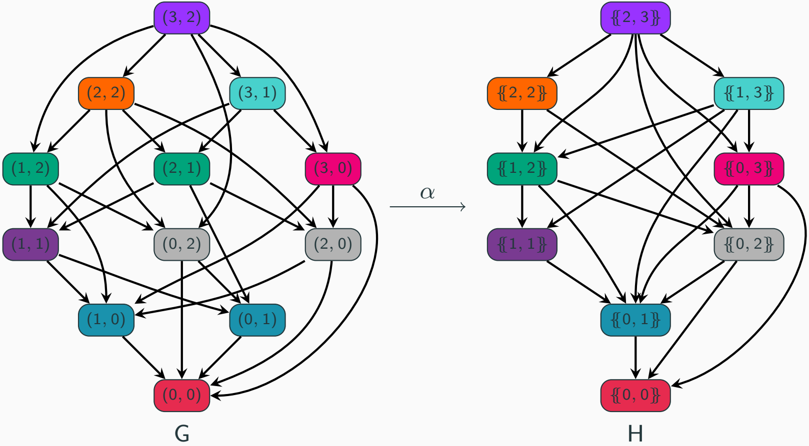
We call a ruleset R slim if every congruence relation on $V(\mathrm{R})$ is trivial.

Proposition

- A ruleset R is slim if and only if every option preserving map from R is injective.
- A ruleset R is slim if and only if the Opt map is injective.

Example

Return to an earlier example. The game on right is slim while the game on the left is not.



Slimification

Proposition

For every ruleset R, there is a unique congruence relation $\bowtie$ on $V(\mathsf{R})$ such that $\mathsf{R}/\bowtie$ is slim.

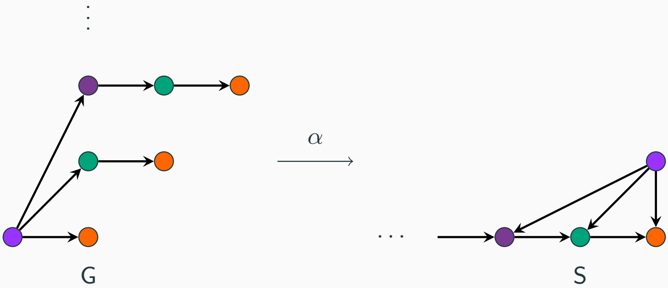
$\mathsf{R}/\bowtie$ is called the slimification of R.

Remark

First Isomorphism Theorem implies that for every ruleset R, there is a surjective option preserving ruleset map $\alpha : \mathsf{R} \to \mathsf{S}$ to a unique (up to isomorphism) slim ruleset S.

Example

An option and source preserving map from a game G to its slimification S.

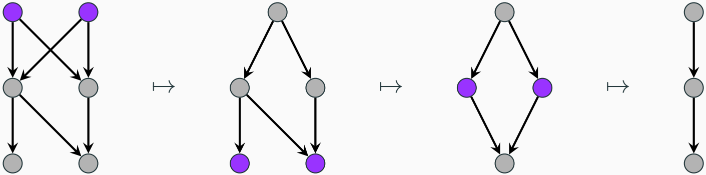


Slimification (continued)

In practice we can find $\bowtie$ using recursion by identifying positions that have the same options until no two such position exist. This process is independent of the order in which one chooses to identify compatible positions.

Example

Slimification process through identifications indicated in purple.



But wait...there's more!

- We have also spelled out the connection between traditionally defined games and our slim games.
- We also have a few enumeration results for slim games and slim rulesets.
- We can (likely) extend our set up and results to partizan rulesets and games by introducing colored arrows.