Morphisms of impartial combinatorial games

Virtual Combinatorial Game Theory Seminar

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Disclaimer

Throughout this talk:

Theorem (BEEPS) = “Theorem” (BEEPS)

Impartial Games

- A finite impartial game G is a finite digraph with a unique source but no infinite directed walk. Each vertex is called a position while the unique source is referred to as the starting position. The elements of the set $\mathrm{Opt}(P)$ of out-neighbors of a position P are called the options of P. A position P is called terminal if $\mathrm{Opt}(P) = \emptyset$. We say that Q is a subposition of P if there is a directed walk from P to Q.
- One can verify that each position of G is a subposition of the starting position. Since G has no infinite walks, no position is a proper subposition of itself.

Impartial Games (continued)

- Each position P of a game G determines a game G_P which is the sub-digraph of G induced by the subpositions of P. If P is the initial position, then G_P is of course G. Replacing each position P in G with the game G_P results in a digraph that we refer to as the game digraph.
- Accordingly, we define

$$\mathrm{Opt}(\mathrm{G}_P) := \{\mathrm{G}_Q \mid Q \in \mathrm{Opt}(P)\}.$$

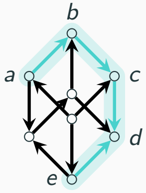
- The nim-number $\mathrm{nim}(P)$ of a position P of a game is defined recursively as the minimum excludant of the nim-numbers of the options of P. That is,

$$\mathrm{nim}(P) := \mathrm{mex}(\mathrm{nim}(\mathrm{Opt}(P))).$$

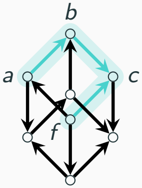
  The nim-number of the game is the nim-number of the starting position.

Digraph Homomorphisms

- For a subset $S \subseteq V(G)$ of a digraph G, we define the induced subgraph $\langle S \rangle$ to be the graph whose vertex set is S and whose edge set consists of all of the directed edges in $E(G)$ that have endpoints in S.



Induced subgraph



Not induced subgraph

- A digraph homomorphism from a digraph G to a digraph H is a map $f : V(G) \to V(H)$ such that if $(u, v) \in E(G)$, then $(f(u), f(v)) \in E(H)$. We simply write $f : G \to H$.
- If $f(G)$ is an induced subgraph of H, then f is called faithful.
- If f is a faithful bijective graph homomorphism, then it is an isomorphism. **Note:** Faithful really is necessary, as not all graph bimorphisms are isomorphisms.

Digraph Homomorphisms (continued)

- Let G be a digraph and let $\mathcal{P} = \{V_1, \ldots, V_k\}$ be a partition of the vertex set of G into nonempty blocks. The quotient graph $G/\mathcal{P}$ of G by $\mathcal{P}$ is the graph whose vertices are the sets $V_1, \ldots, V_k$ and whose directed edges are the pairs (V_i, V_j) for $i \neq j$, such that there exist $u_i \in V_i$, $v_j \in V_j$ with $(u_i, u_j) \in E(G)$.
- Put another way, a quotient graph Q of a graph G is a graph whose vertices are blocks of a partition of the vertices of G and where block B is adjacent to block C if some vertex in B is adjacent to some vertex in C.

Digraph Homomorphisms (continued)

A graph homomorphism $f : G \to H$ gives rise to an equivalence relation $\equiv_f$, called the kernel of f, defined on $V(G)$ by $u \equiv_f v$ if and only if $f(u) = f(v)$. (This works for any function!) This induces a partition $\mathcal{P}_f$ on the vertex set of G. We write G/f for $G/\mathcal{P}_f$ and say that we are taking the quotient of G by f.

The following result can be thought of as the analog to the Fundamental Homomorphism Theorem (aka, 1st Isomorphism Theorem) for algebraic structures.

Theorem

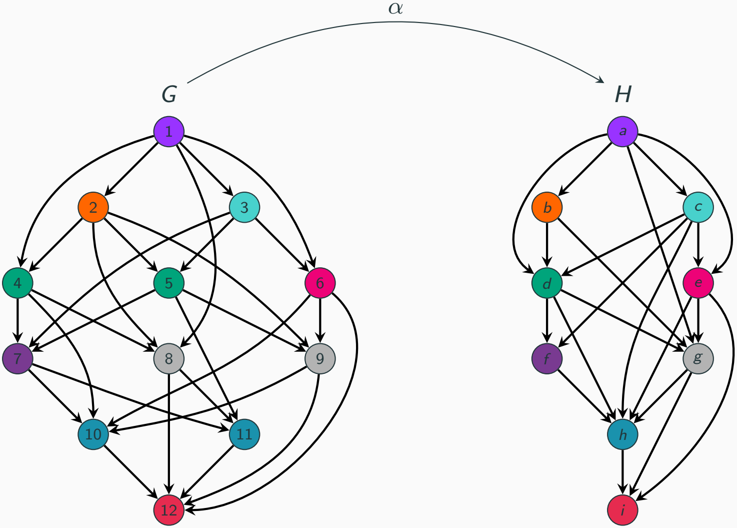
If $f : G \to H$ is a faithful graph homomorphism, then the image of f is isomorphic to the quotient graph G/f.

- Two games are isomorphic if their corresponding digraphs are isomorphic.
- For games G and H, if $\alpha : G \to H$ satisfies $\mathrm{Opt}(\alpha(a)) = \alpha(\mathrm{Opt}(a))$ for each position a of G, then α is option preserving. If α takes the starting position of G to the starting position of H, then α is called source preserving. If α is both option and source preserving, then α is a game morphism.
- Certainly, not every digraph homomorphism is option preserving.

Theorem (BEEPS)

For an option preserving map $\alpha : G \to H$, α is source preserving iff α is surjective.

The following mapping $\alpha : G \to H$ determined by matching colors is both option preserving and source preserving, and hence a game morphism.



Game Morphisms (continued)

Theorem (BEEPS)

If $\alpha : G \to H$ is option preserving, then α is a faithful digraph homomorphism.

Note

- If $\alpha : G \to H$ is an option preserving map, then each equivalence class that arises from the kernel of α will be referred to as a position class.
- In an upcoming paper, Bašić et al. define a good partition and what it means for two games to emulationally equivalent. Definitions omitted here.

Theorem (BEEPS)

If $\alpha : G \to H$ is an option preserving map, then the partition consisting of the position classes is good.

Theorem (BEEPS)

Every good partition of an impartial game G determines an option preserving map.

Game Morphisms (continued)

Theorem (BEEPS)

Two games G and H are emulationally equivalent if and only if there are game morphisms $G \to K$ and $H \to K$ for some game K.

Theorem (BEEPS)

If $\alpha : G \to H$ is option preserving, then for each position a in G:

(a) $\text{nim}(a) = \text{nim}(\alpha(a))$;

(b) a and $\alpha(a)$ have same birthday.

Fundamental Game Morphism Theorem

The following result can be thought of as the First Isomorphism Theorem for impartial games.

Theorem (BEEPS)

If $\alpha : G \to H$ is option preserving, then the image of α is isomorphic to the quotient graph G/α.

Simple Games

Mimicking the idea of simple groups, we can call a game G simple if every option preserving map from G is injective. Equivalently, G is simple if every good partition is trivial.

Theorem (BEEPS)

A game G is simple iff the Opt map is injective (i.e., no two different positions have exactly the same options).

Example

In our earlier example, the game H is simple while the game G is not.

Simple Games (continued)

Theorem (BEEPS)

For any game G, there is game morphism $\alpha : G \to S$ to a unique (up to isomorphism) simple game S. We call S the reduction of G.

Example

In our earlier example, the game H is the reduction of G.

Corollary (BEEPS)

Two games are emulationally equivalent iff their reductions are isomorphic.

Big Picture

Emulational equivalence is an equivalence relation on the class of games and the simple games form a class of unique (up to isomorphism) representatives.

To Do List

- Did we make the right categorical choice? There are three natural choices:
  1. Objects are games, morphisms are option preserving maps (source not necessarily sent to source).
  2. Objects are games, morphisms are option preserving and source preserving maps.
  3. Objects are rulesets, morphims are option preserving maps.
- Verify all the claims we just made!
- Enumerate simple games by either number of vertices or by birthday.

Thank you!