REVIEW: Algebra, Trigonometry, Functions (domain, arithmetic, composition, inverse)

1. A camera is mounted at a point on the ground 500 meters from the base of a space shuttle launching pad. The shuttle rises vertically when launched, and the camera's angle of elevation is continually adjusted to follow the bottom of the shuttle. Express the height x as function of the angle of elevation θ.

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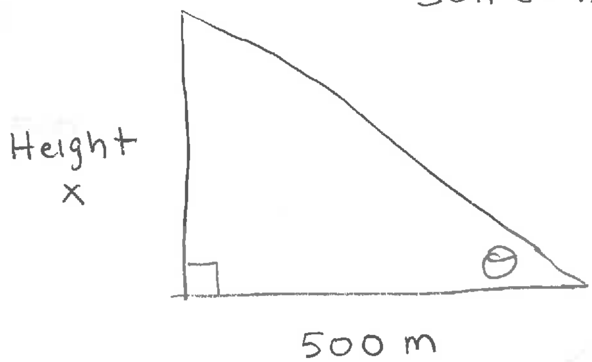
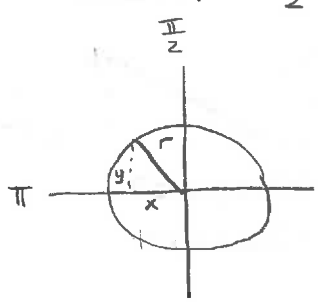


$\tan(\theta) = \frac{x}{500}$

$\rightarrow \boxed{x = 500\tan(\theta)}$

2. Given that $\cot(\theta) = -\frac{3}{2}$ for $\pi/2 \le \theta \le \pi$, find the exact value of each of the five remaining trig functions at θ.

$\cot(\theta) = -\frac{3}{2} = \frac{x}{y} \rightarrow r = \sqrt{x^2+y^2} = \sqrt{(-3)^2+(2)^2} = \sqrt{13}$



Thus $\sin(\theta) = \frac{y}{r} = \frac{2}{\sqrt{13}}$

$\cos(\theta) = \frac{x}{r} = \frac{-3}{\sqrt{13}}$

$\tan(\theta) = \frac{y}{x} = -\frac{2}{3}$

$\csc(\theta) = \frac{r}{y} = \frac{\sqrt{13}}{2}$

$\sec(\theta) = \frac{r}{x} = \frac{\sqrt{13}}{-3}$

3. Simplify and express all powers in terms of positive exponents $\sqrt[3]{\frac{-8x^5y^{-8}}{x^{-1}y^4}}$.

$$\sqrt[3]{\frac{-8x^5y^{-8}}{x^{-1}y^4}} = \left(\frac{-8x^6}{y^{12}}\right)^{1/3} = \boxed{-\frac{2x^2}{y^4}}$$

4. Write the expression as a single logarithm: $\ln 2 + 5\ln x^2 - \frac{1}{2}\ln y$.

$\ln 2 + 5\ln x^2 - \frac{1}{2}\ln y$

$= \ln 2 + \ln(x^2)^5 - \ln\sqrt{y}$

$= \ln(2\cdot x^{10}) - \ln\sqrt{y}$

$= \boxed{\ln\left(\frac{2x^{10}}{\sqrt{y}}\right)}$

5. Write the expression as a sum or difference of logarithms: $\ln\sqrt{\frac{x^3y^4}{3z}}$.

$$\ln\sqrt{\left(\frac{x^3y^4}{3z}\right)} = \ln\left(\frac{x^3y^4}{3z}\right)^{1/2}$$
$$= \frac{1}{2}\ln\left(\frac{x^3y^4}{3z}\right)$$
$$= \frac{1}{2}\ln(x^3) + \frac{1}{2}\ln(y^4) - \frac{1}{2}\ln(3z) = \cancel{\boxed{\frac{3}{2}\ln(x) + 2\ln(y) - \frac{3}{2}\ln(z)}}$$
$$= \frac{3}{2}\ln x + 2\ln y - \frac{1}{2}\ln 3 - \frac{1}{2}\ln z$$

6. Combine the fractions over a common denominator and simplify: $\frac{1}{x^2-3x} + \frac{2x}{x^2-9}$.

$$\frac{1}{x^2-3x} + \frac{2x}{x^2-9} = \frac{1}{x(x-3)} + \frac{2x}{(x+3)(x-3)}$$
$$\rightarrow \frac{(x+3)}{x(x+3)(x-3)} + \frac{2x^2}{x(x+3)(x-3)} = \boxed{\frac{2x^2+x+3}{x(x+3)(x-3)}}$$

7. Compute $e^{\ln 2 + \ln\frac{1}{2}}$.

$$e^{\ln 2 + \ln 1/2} = e^{\ln 2} \cdot e^{\ln 1/2} = 2\left(\frac{1}{2}\right) = \boxed{1}$$

8. Simplify in terms of $\cos x$ and $\sin x$: $\frac{\sec x \cos^2 x \tan x}{\sin x \cot^2 x \csc^3 x}$

$$\frac{\sec x \cos^2 x \tan x}{\sin x \cot^2 x \csc^3 x} = \frac{\frac{1}{\cancel{\cos x}} \cdot \cos^2 x \cdot \frac{\sin x}{\cancel{\cos x}}}{\cancel{\sin x} \cdot \frac{\cos^2 x}{\sin^2 x} \cdot \frac{1}{\sin^3 x}} = \frac{\sin x}{\frac{\cos^2 x}{\sin^4 x}} = \boxed{\frac{\sin^5 x}{\cos^2 x}}$$

9. Without using a calculator or computer, evaluate the following:

(a) $\cos 210°$

$\frac{\sqrt{3}}{2}$

(b) $\sin\frac{\pi}{6}$

$\frac{1}{2}$

(c) $\tan^{-1}(1)$

$\frac{\pi}{4}$ OR $\frac{5\pi}{4}$

(d) $\sin^{-1}(0)$

0 OR π OR 2π

(e) $\cos^{-1}(-1)$

π

(f) $\tan^{-1}(\sqrt{3})$

$\frac{\pi}{3}$ OR $\frac{4\pi}{3}$

(g) $\cos\frac{\pi}{3}$

$\frac{1}{2}$

(h) $\tan\frac{3\pi}{4}$

-1

10. Solve for x without using a calculator or computer: $\log_2 x = 3$

$\log_2(x) = 3 \iff x = 2^3 = 8$

11. Solve for x without using a calculator or computer: $\log_3 x^2 = 2\log_3 4 - 4\log_3 5$

$\log_3 x^2 = 2\log_3 4 - 4\log_3 5$

$\rightarrow \log_3 x^2 = \log_3\left(\frac{4^2}{5^4}\right)$

$\rightarrow x^2 = \frac{4^2}{5^4} \rightarrow \boxed{x = \pm\frac{4}{25}}$

12. Determine an equation of the line through (-1,3) and (2,-4).

$m = \text{Slope} = \frac{-4-3}{2-(-1)} = -\frac{7}{3}$

$\rightarrow \boxed{\cancel{m = \frac{1}{3}}}$

$y - 3 = -\frac{7}{3}(x-(-1))$

$\rightarrow y - 3 = -\frac{7}{3}x + -\frac{7}{3}$

$\rightarrow \boxed{y = -\frac{7}{3}x + \frac{2}{3}}$

13. Determine an equation of the line through (-1,2) and perpendicular to the line $2x - 3y + 5 = 0$.

$2x - 3y + 5 = 0$

$\rightarrow y = \frac{2}{3}x + \frac{5}{3}$

Slope of line $= \frac{2}{3}$

Thus slope of line perpendicular to $2x - 3y + 5 = 0$ is $-\frac{3}{2}$.

EQ: $y - 2 = -\frac{3}{2}(x-(-1))$

$\rightarrow y - 2 = -\frac{3}{2}x - \frac{3}{2}$

$\rightarrow \boxed{y = -\frac{3}{2}x + \frac{1}{2}}$

14. Find the domain of the function $f(x) = \dfrac{1}{\sqrt{x^2 - x - 2}}$.

→ Denominator can't = zero

→ Can't have negative under $\sqrt{\ }$

→ Thus we need $x^2 - x - 2 > 0$

Need $x^2 - x - 2 = (x-2)(x+1) > 0$

(1) $(x-2) > 0$ and $(x+1) > 0$

$\rightarrow x > 2$ and $x > -1$

Thus $x \in (2, \infty)$

OR

(2) $(x-2) < 0$, $(x+1) < 0$

$\rightarrow x < 2$ and $x < -1$

Thus $x \in (-\infty, -1)$

Therefore our domain is $x \in (-\infty, -1) \cup (2, \infty)$

15. Find the domain and range of the functions: (a) $f(x) = 7$, and (b) $g(x) = \dfrac{5x-3}{2x+1}$

(a) Domain $= \mathbb{R}$
Range $= 7$

(b) Need denominator to not equal zero

Thus $2x + 1 = 0$ when $x = -\frac{1}{2}$.

Therefore our domain is all $\mathbb{R}$ except $-1/2$
ie
$x \in (-\infty, -1/2) \cup (-1/2, \infty)$

Range $y = \frac{5x-3}{2x+1}$

$\rightarrow y(2x+1) = 5x - 3$

$\rightarrow 2xy + y - 5x + 3 = 0$

$\rightarrow 2xy - 5x = -(y+3)$

$\rightarrow x(2y-5) = -(y+3)$

$\rightarrow x = -\frac{(y+3)}{2y-5}$ Thus $2y - 5 \neq 0$

$\rightarrow y \neq \frac{5}{2}$

Therefore Range is all $\mathbb{R}$ except $\frac{5}{2}$
ie $y \in (-\infty, \frac{5}{2}) \cup (\frac{5}{2}, \infty)$

Key for pg. 3

17. square

$A_s = (2r)^2 = 4r^2$

Circle

$A_c = \pi r^2$

Area Square - Circle

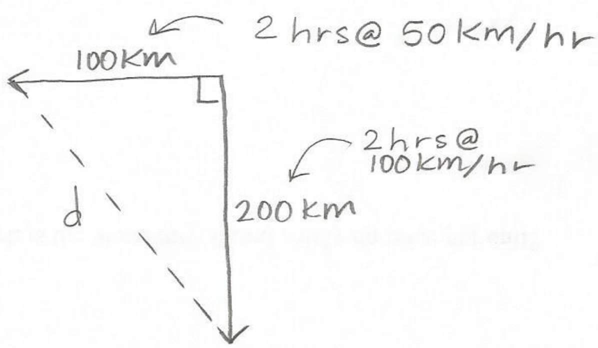
$A_{s-c} = 4r^2 - \pi r^2$

Ratio: $\frac{4r^2 - \pi r^2}{4r^2} = \frac{4-\pi}{r}$

18. $P = \underbrace{\frac{1}{2}(2\pi r)}_{\text{semi-circle}} + \underbrace{r + r}_{\text{sides}} + \underbrace{2r}_{\text{bottom}}$

$= \pi r + 4r$

19.



$(100)^2 + (200)^2 = d^2$

$10000 + 40000 = d^2$

$\sqrt{50000} = d$

$100\sqrt{5}$ km = distance

20 false.

Let $f(x) = x+1$ and $g(x) = x^2$.

Then $f \circ g = x^2+1$ and $g \circ f = (x+1)^2 = x^2+2x+1$

21. $f(x) = 8 - x$

$g(x) = 3x^2 - x + 4$

[Another example of why #21 is false!]

$$(g \circ f)(x) = 3(8-x)^2 - (8-x) + 4$$
$$= 3(64 - 16x + x^2) - 8 + x + 4$$
$$= 192 - 48x + 3x^2 - 8 + x + 4$$
$$= 3x^2 - 47x + 188$$

$$(f \circ g)(x) = 8 - (3x^2 - x + 4)$$
$$= 8 - 3x^2 + x - 4$$
$$= -3x^2 + x + 4$$

22. a) Yes, this scenario could define a function. Each member of the domain (the set of 10 missiles) hits one and only one member of the range (the set of targets).

b) No, this scenario does not define a function. The domain (Google search phrase) is mapped to multiple members of the range (return entries).

23.

$$x = \frac{y+2}{5y-1}$$

$$(5y-1)x = (5y-1)\left(\frac{y+2}{5y-1}\right)$$

$$5xy - x = y + 2$$
$$5xy - y = x + 2$$
$$y(5x-1) = x+2$$

$$y = \frac{x+2}{5x-1}$$

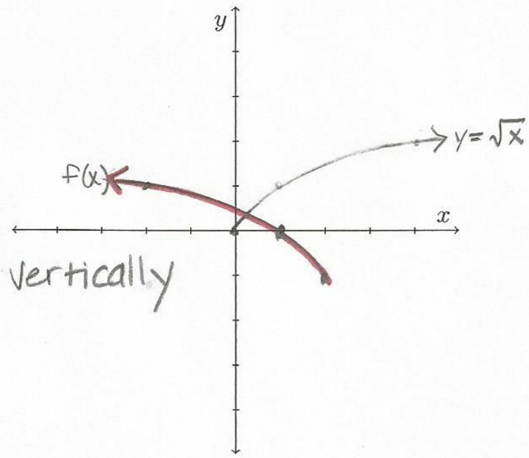
$$f^{-1}(x) = \frac{x+2}{5x-1}$$

Graphing and function transformations

24 Sketch the graph of each function without using a calculator. In each case, identify how you obtained the graph from the graph of the function in parentheses.

(a) $f(x) = -1 + \sqrt{2-x}$ $\quad (y = \sqrt{x})$

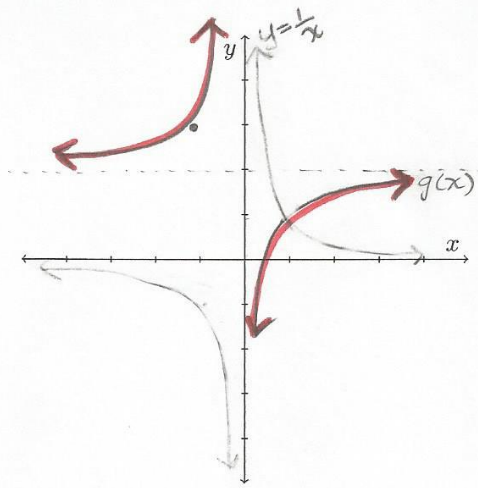
* f(x) is the graph of $y=\sqrt{x}$ reflected over the y-axis and then shifted horizontally 2 units to the right and vertically one unit down.



(b) $g(x) = \dfrac{2x-1}{x}$ $\quad (y = \dfrac{1}{x}$; Hint: first, split into two fractions.)

$g(x) = \frac{2x}{x} - \frac{1}{x}$

$g(x) = 2 - \frac{1}{x}$



* g(x) is the graph of $y=\frac{1}{x}$ reflected over the x-axis and shifted 2 units up.

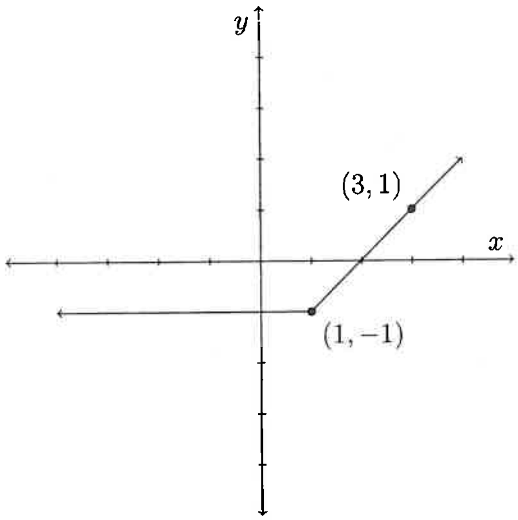
25 The graph of $y = a\dfrac{1}{\cos(x+b)+2} + c$ results when the graph of $y = \dfrac{1}{\cos(x)+2}$ is reflected over the x-axis, shifted 3 units to the right, and then shifted 4 units down. Find a, b, and c.

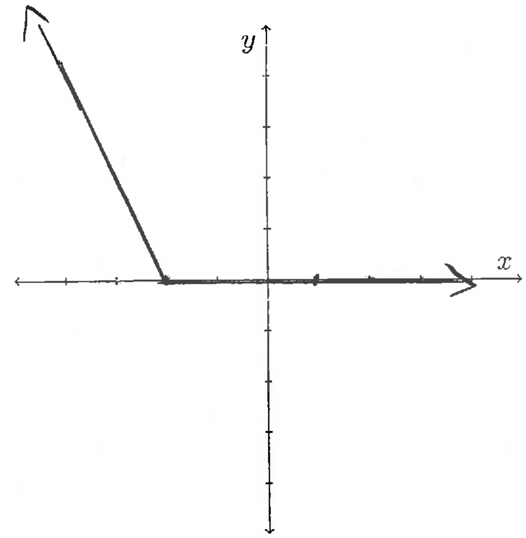
Reflected over x axis: $a = -1$

Shifted 3 units right: $b = -3$

Shifted 4 units down: $c = -4$

26 Consider the graph of the function $y = f(x)$ given in the left figure below. Using the axes provided on the right, sketch the graph of the function $y = 2f(-1-x)+2$.





27 Consider the function $f(x) = -\sqrt{2(x+1)} - 3$. How would you obtain the graph of f from the function $g(x) = \sqrt{x}$? That is, describe in words the sequence (order matters) of transformations for obtaining the graph of f from the graph of g. You do *not* need to sketch the graph of either function.

vertically shift down 3, horizontally shift to the left 2, horizontally shrink by a factor of 2, reflect the function across the x-axis

28 Find the equation of the parabola that has vertex $(2,-1)$ and passes through the point $(-1,6)$. *Hint:* A useful form for a parabola is $y = a(x-h)^2 + k$, where a, h, and k are fixed real numbers.

$6 = a(-1-2)^2 + -1$

$6 = a(-3)^2 - 1$

$6 = 9a - 1$

$7 = 9a$

$7/9 = a$

$y = 7/9\,(x-2)^2 - 1$

Rate of change

29 Suppose a ball is thrown off a 100 foot tall building such that the height of the ball in feet at time t in seconds is given by $h(t) = -16t^2 + 25t + 100$.

(a) What is average rate of change over the first second of flight?

$$\frac{h(0)-h(1)}{0-1} = \frac{100-(-16+25+100)}{-1} = \frac{-9}{-1} = 9\text{ ft/sec}$$

(b) How about over $[0, 2]$? Interpret the sign of your answer.

$$\frac{h(0)-h(2)}{0-2} = \frac{100-(-16(4)+25(2)+100)}{-2} = \frac{100-86}{-2} = \frac{14}{-2} = -7\text{ ft/sec}$$

the ball is traveling downward at 7ft/sec

30 The position in meters of a particle moving in a straight line is given for some values of time t in seconds in the following table.

t	0	.1	.2	.3	.4
$p(t)$	0	.5	.7	1.2	3

(a) What is the average velocity over the first .3 seconds of movement?

$$\frac{1.2-0}{.3-0} = 4\text{ m/sec}$$

(b) Estimate the instantaneous velocity at $t = .2$ seconds.

$0.1 \leq t \leq 0.2$

$$\frac{0.7-0.5}{0.2-0.1} = \frac{0.2}{0.1} = 2\text{ m/s}$$

$0.2 \leq t \leq 0.3$

$$\frac{1.2-0.7}{0.3-0.2} = \frac{0.5}{0.1} = 5\text{ m/s}$$

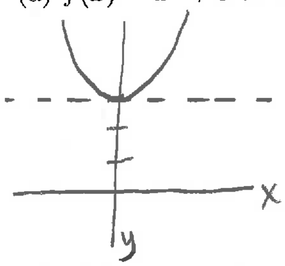
estimate for instantaneous velocity at $t = 0.2$ secs

$$\tfrac{1}{2}(2+5) = \tfrac{7}{2}\text{ m/sec}$$

Intuitive derivative

31 Graph each function below and then sketch the tangent line to that function (using a dashed line) at the given point. Then use the dashed line and state the derivative of the function at the given point. Do not use any shortcuts you may know for the derivative.

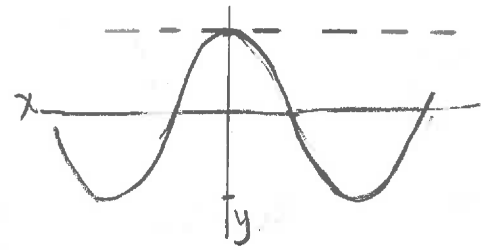
(a) $f(x) = x^2 + 3$ at $x = 0$



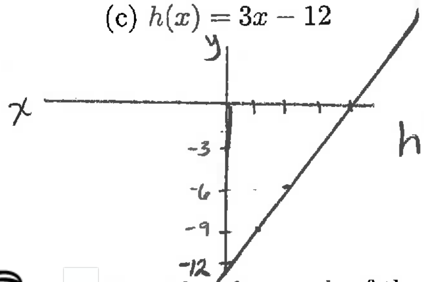
$f'(x) = 0$

(b) $g(x) = \cos x$ at $x = 0$

$g'(x) = 0$

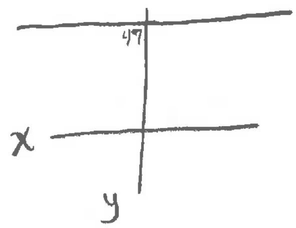


(c) $h(x) = 3x - 12$



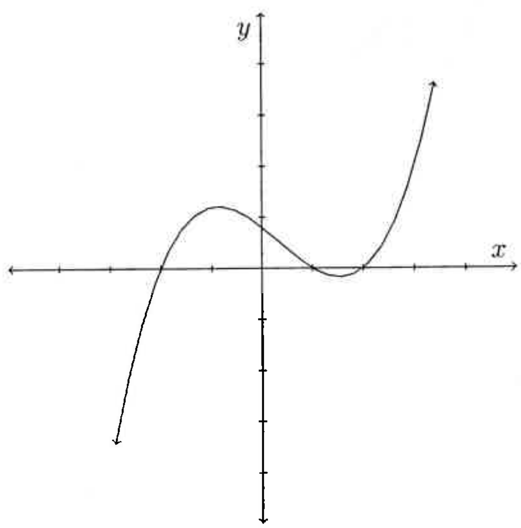
$h'(x) = 3$

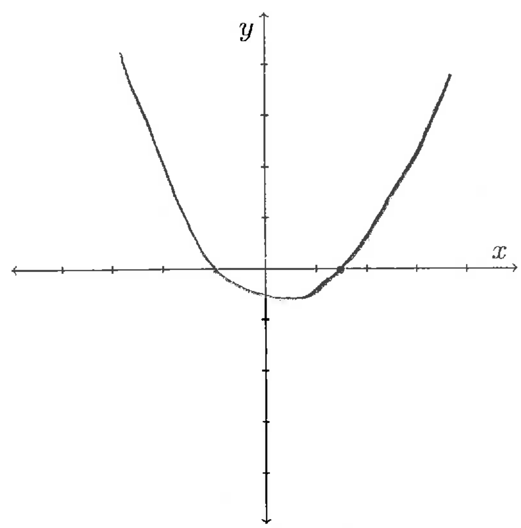
(d) $f(x) = 47$



$f'(x) = 0$

32 Consider the graph of the function $y = f(x)$ given in the left figure below.





(a) Using the axes provided on the right, sketch the graph of the derivative $y = f'(x)$.

(b) Put the following expressions in increasing order: $f'(-2), f'(-1), f'(0), f'(2)$.

$f'(0), f'(-1), f'(2), f'(-2)$

Limits

33 True or False? Justify your answer.

(a) If a function $f(x)$ does not have a limit as x approaches a from the left, then $f(x)$ does not have a limit as x approaches a from the right.

False: The function $f(x) = \begin{cases} 1/x & x<0 \\ x & x \geq 0 \end{cases}$ has $\lim_{0^-} f$ DNE but $\lim_{0^+} f = 0$

(b) If $h(x) \leq f(x) \leq g(x)$ for all real numbers x and $\lim_{x\to a} h(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x)$ also exists.

False. If $h(x) = x^2+1$, $g(x) = -x^2-1$ and $f(x) = \sin(\frac{1}{x})$ then $\lim_0 h = 1$, $\lim_0 g = -1$ (so both exist) and $h(x) \leq f(x) \leq g(x)$ but $\lim_0 f$ DNE

34 Given the graph of f (assume f continues beyond the box) find the following: If the limit does not exist also state why it does not exist.

(a) $\lim_{x\to 1^-} f =$ 2

(b) $\lim_{x\to 1+} f =$ 2

(c) $\lim_{x\to 1} f =$ 2

(d) $\lim_{x\to 2} f(x) =$ 2

(e) $\lim_{x\to 3} f =$ DNE

(f) $f(3) =$ 1

(g) $\lim_{x\to 4} f =$ 1

(h) $\lim_{x\to -1^-} f =$ 1

(i) $\lim_{x\to -1^+} f(x) =$ 2

(j) $\lim_{x\to -1} f =$ DNE

(k) $\lim_{x\to \infty} f(x) =$ 0

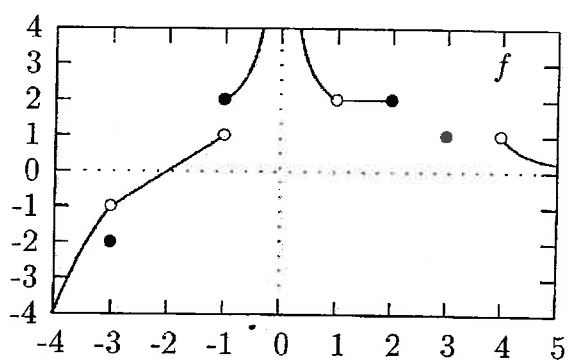
(l) $\lim_{x\to -\infty} f = -\infty$

(m) $\lim_{x\to -4} f = -4$

(n) $\lim_{x\to -3} f = -1$

(o) $f(-3) = -2$

(p) $\lim_{x\to 0} f = \infty$



35 Evaluate each of the following limits. If a limit does not exist, specify whether the limit equals ∞, $-\infty$, or simply does not exist (in which case, write DNE).

(a) $\lim\limits_{x\to 2} x^2 + 4x - 12$

$= 2^2 + 4\cdot 2 - 12$

$= 4 + 8 - 12 = \boxed{0}$

(b) $\lim\limits_{x\to 2} \dfrac{x^2+4x-12}{x^2+4x+3}$

$= \dfrac{4+8-12}{4+8+3} = \dfrac{0}{15} = \boxed{0}$

(c) $\lim\limits_{x\to 2} \dfrac{x^2+4x-12}{x^2-x-2}$ ("$\frac{0}{0}$" form)

$= \lim\limits_{x\to 2} \dfrac{(x+6)\cancel{(x-2)}}{\cancel{(x-2)}(x+1)} = \lim\limits_{x\to 2} \dfrac{x+6}{x+1}$

$= \boxed{\dfrac{8}{3}}$

(d) $\lim\limits_{x\to 2} \dfrac{x^2+4x-12}{x^2-4x+4}$ ("$\frac{0}{0}$" form)

$= \lim\limits_{x\to 2} \dfrac{(x+6)\cancel{(x-2)}}{(x-2)\cancel{(x-2)}} = \lim\limits_{x\to 2} \left(\dfrac{x+6}{x-2}\right)$

$\boxed{\text{DNE}}$

(e) $\lim\limits_{x\to 4} \dfrac{x-4}{\sqrt{x}-2}$ ("$\frac{0}{0}$" form)

$= \lim\limits_{x\to 4} \dfrac{\cancel{(\sqrt{x}-2)}(\sqrt{x}+2)}{\cancel{(\sqrt{x}-2)}} = 2+2 = \boxed{4}$

(f) $\lim\limits_{x\to 3} \dfrac{1}{x-3}$

DNE

(g) $\lim\limits_{x\to 3} \dfrac{1}{(x-3)^2}$ $= \boxed{\infty}$

(h) $\lim\limits_{x\to \pi} \dfrac{x}{\cos x} = \dfrac{\pi}{-1} = \boxed{-\pi}$

(i) $\lim\limits_{x\to 0} \dfrac{\frac{1}{x+4}-\frac{1}{4}}{x} = \lim\limits_{x\to 0} \dfrac{\frac{4-x-4}{4(x+4)}}{x}$

$= \lim\limits_{x\to 0} \dfrac{-x}{4(x+4)} \cdot \dfrac{1}{x}$

$= -\dfrac{1}{4\cdot 4} = \boxed{-\dfrac{1}{16}}$

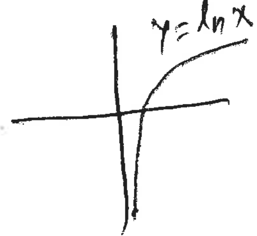
(j) $\lim\limits_{x\to 5} \dfrac{|x-5|}{x-5}$

$\lim\limits_{x\to 5^-} \dfrac{|x-5|}{x-5} = \lim\limits_{x\to 5^-} \dfrac{-(x-5)}{x-5} = -1$

$\lim\limits_{x\to 5^+} \dfrac{|x-5|}{x-5} = \lim\limits_{x\to 5^+} \dfrac{x-5}{x-5} = 1$

DNE.

(k) $\lim\limits_{x\to 0^+} \ln x$ (hint: graph it)



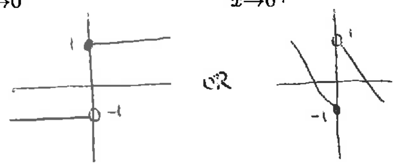
From graph.

$\lim\limits_{x\to 0^+} \ln x = \boxed{-\infty}$

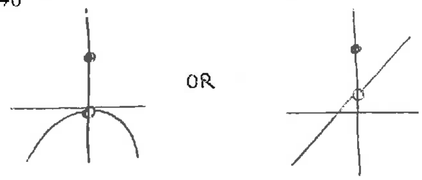
(l) $\lim\limits_{x\to 0} \ln x$ (hint: graph it) $\boxed{-\infty}$

36 Sketch the graphs of possible functions g, and h such that: g satisfies property (a) below and h satisfies property (b) below. (There should be two separate graphs.)

(a) $\lim_{x\to 0^-} g(x) = -1$ and $\lim_{x\to 0^+} g(x) = +1$



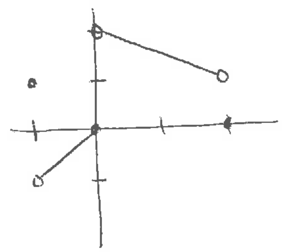
(b) $\lim_{x\to 0} h(x) \neq h(0)$, where $h(0)$ is defined.



(there are many possible options)

37 Sketch the graph of a possible function f that has all properties (a)–(g) listed below.

(a) The domain of f is $[-1, 2]$
(b) $f(0) = f(2) = 0$
(c) $f(-1) = 1$
(d) $\lim_{x\to 0^-} f(x) = 0$
(e) $\lim_{x\to 0^+} f(x) = 2$
(f) $\lim_{x\to 2^-} f(x) = 1$
(g) $\lim_{x\to -1^+} f(x) = -1$



38 Let f and g be functions such that $\lim_{x\to a} f(x) = -3$ and $\lim_{x\to a} g(x) = 6$. Evaluate the following limits, if they exist.

(a) $\lim_{x\to a} \frac{(g(x))^2}{f(x)+5}$ $= \frac{\lim_{x\to a}(g(x))^2}{\lim_{x\to a}(f(x)+5)} = \frac{(\lim_{x\to a} g(x))^2}{\lim_{x\to a} f(x) + \lim_{x\to a} 5} = \frac{(6)^2}{-3+5} = \frac{36}{2} = \boxed{18}$

(b) $\lim_{x\to a} \frac{7f(x)}{2f(x)+g(x)}$ $= \frac{\lim_{x\to a} 7f(x)}{\lim_{x\to a}(2f(x)+g(x))} = \frac{7\lim_{x\to a} f(x)}{\lim_{x\to a} 2f(x) + \lim_{x\to a} g(x)} = \frac{7\lim_{x\to a} f(x)}{2\lim_{x\to a} f(x) + \lim_{x\to a} g(x)} = \frac{7(-3)}{2(-3)+6} = \frac{-21}{0} \Rightarrow$ DNE

(c) $\lim_{x\to a} \sqrt[3]{g(x)+2}$ $= \sqrt[3]{\lim_{x\to a}(g(x)+2)} = \sqrt[3]{\lim_{x\to a} g(x) + \lim_{x\to a} 2} = \sqrt[3]{6+2} = \sqrt[3]{8} = \boxed{2}$

39

Let f be defined as follows.

$$f(x) = \begin{cases} 3x & \text{if } x < 0 \\ 3x + 4 & \text{if } 0 \le x \le 4 \\ x^2 & \text{if } x > 4 \end{cases}$$

For (a)–(h), evaluate the limits if they exist. If a limit does not exist, specify whether the limit equals ∞, $-\infty$, or simply does not exist (in which case, write DNE). For part (i), answer the question.

(a) $\lim_{x \to 0^+} f(x)$ $3(0)+4 = \boxed{4}$

(b) $\lim_{x \to 0^-} f(x)$ $3(0) = \boxed{0}$

(c) $\lim_{x \to 0} f(x)$ $\boxed{\text{DNE}}$

(d) $f(0)$ $3(0)+4 = \boxed{4}$

(e) $\lim_{x \to 4^+} f(x)$ $(4)^2 = \boxed{16}$

(f) $\lim_{x \to 4^-} f(x)$ $3(4)+4 = \boxed{16}$

(g) $\lim_{x \to 4} f(x)$ $\boxed{16}$

(h) $f(4)$ $3(4)+4 = \boxed{16}$

(i) Determine where f is continuous.

$(-\infty, 0), (0, \infty)$

40

If $3x \le f(x) \le x^3 + 2$ for $0 \le x \le 2$, evaluate $\lim_{x \to 1} f(x)$.

$\lim_{x \to 1} 3x = 3(1) = 3$ $\lim_{x \to 1} x^3+2 = 1^3+2 = 3$

So, by the Squeeze Theorem $\boxed{\lim_{x \to 1} f(x) = 3}$

41

Use the Squeeze Theorem to prove that $\lim_{x \to 0} x^4 \cos(2/x) = 0$.

The value of $\cos\left(\frac{2}{x}\right)$ is always between 1 and -1 because $-1 \le \cos(x) \le 1$.

Thus $-x^4 \le x^4 \cos(2/x) \le x^4$.

$\lim_{x \to 0} -x^4 = -(0)^4 = 0$, $\lim_{x \to 0} x^4 = 0^4 = 0$

So, by the Squeeze Theorem, $\lim_{x \to 0} x^4 \cos(2x) = 0$. $\square$

Continuity

42 True or False? (Justify your answer) If a function is not continuous at $x = a$, then either it is not defined at $x = a$ or it does not have a limit as x approaches a.

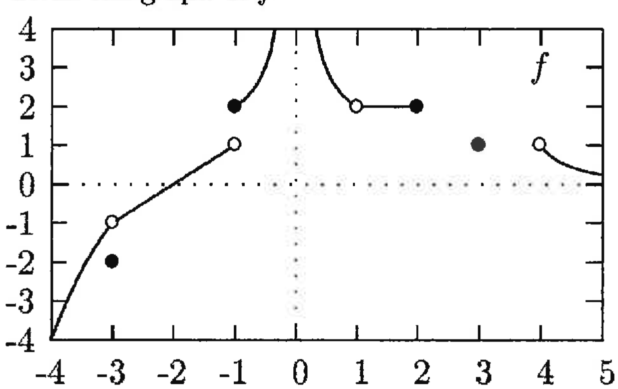
The statement is false. Consider $f(x) = \begin{cases} x, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$

Then $\lim_{x\to 0} f(x) = 0$ while $f(0) = 1$. Thus, f is not cont at $x = 0$.

43 Provide an example of function that is continuous everywhere but does not have a tangent line at $x = 0$. Explain your answer.

Consider $f(x) = |x|$. Then f is cont on $(-\infty, \infty)$, but f does not have a tangent line at $x = 0$.

44 Given the graph of f



State every integer point(s) in the domain of f where f (restricted to its domain) is discontinuous and state why.

$x = -3$: $\lim_{x\to -3} f(x) \neq f(-3)$

$x = -1$: $\lim_{x\to -1} f(x)$ DNE

Note: For all $x \in \{1\} \cup (2,3) \cup (3,4]$, f is not cont at x. However, these values are not in the domain of f.

Definition of derivative

45 Use the definition of the derivative to find the derivative of the following functions

(a) $f(x) = x^2 + 16x - 57$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + 16(x+h) - 57 - (x^2 + 16x - 57)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{16x} + 16h \cancel{- 57} \cancel{- x^2} \cancel{- 16x} \cancel{+ 57}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h + 16)}{\cancel{h}} = \boxed{2x + 16}$$

(b) $f(x) = \sqrt{5x - 17}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{5(x+h) - 17} - \sqrt{5x - 17}}{h} \cdot \frac{\sqrt{5(x+h) - 17} + \sqrt{5x - 17}}{\sqrt{5(x+h) - 17} + \sqrt{5x - 17}}$$

$$= \lim_{h \to 0} \frac{5(x+h) - 17 - (5x - 17)}{h(\sqrt{5(x+h) - 17} + \sqrt{5x - 17})}$$

$$= \lim_{h \to 0} \frac{\cancel{5x} + 5\cancel{h} \cancel{- 17} \cancel{- 5x} \cancel{+ 17}}{\cancel{h}(\sqrt{5(x+h) - 17} + \sqrt{5x - 17})}$$

$$= \lim_{h \to 0} \frac{5}{\sqrt{5(x+h) - 17} + \sqrt{5x - 17}} = \boxed{\frac{5}{2\sqrt{5x - 17}}}$$

(c) $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{(x+h)x} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x} - \cancel{x} - \cancel{h}}{(x+h)x \cdot \cancel{h}} = \lim_{h \to 0} \frac{-1}{(x+h)x} = \boxed{-\frac{1}{x^2}}$$

(d) $f(x) = \frac{1}{x^2}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{x^2 - (x+h)^2}{(x+h)^2 x^2} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} - \cancel{x^2} - 2xh - h^2}{(x+h)^2 x^2 \cdot h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-2x - h)}{(x+h)^2 x^2 \cdot \cancel{h}} = \frac{-2x}{x^2 \cdot x^2} = \boxed{-\frac{2}{x^3}}$$

Local linearization

46 Let $f(x) = x^2 - x$.

(a) Find the slope of the tangent line at $x = 2$ using the limit definition.

$$f'(2) = \lim_{h\to 0} \frac{[(2+h)^2 - (2+h)] - [2^2 - 2]}{h} = \lim_{h\to 0} \frac{4 + 4h + h^2 - 2 - h - 4 + 2}{h}$$

$$= \lim_{h\to 0} \frac{h(4+h)}{h} = \lim_{h\to 0} 4 + h = \boxed{4}$$

(b) Find the equation of the tangent line to the graph of $f(x)$ at $x = 2$.

$f(2) = 2^2 - 2 = 2$

$y - 2 = 4(x-2)$ or $y = 4(x-2) + 2$

or $y = 4x - 6$

47 Use the definition of the derivative and linearization to approximate $\sqrt{9.2}$.

lets use $f(x) = \sqrt{x}$ and $a = 9$

$$f'(9) = \lim_{h\to 0} \frac{\sqrt{9+h} - \sqrt{9}}{h} \frac{(\sqrt{9+h} + \sqrt{9})}{(\sqrt{9+h} + \sqrt{9})} = \lim_{h\to 0} \frac{9+h-9}{h(\sqrt{9+h} + \sqrt{9})}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{9+h} + \sqrt{9})} = \frac{1}{\sqrt{9} + \sqrt{9}} = \frac{1}{6}$$

$f(9) = \sqrt{9} = 3$ linearization $l_9(x) = 3 + \frac{1}{6}(x-9)$

now plug in 9.2 to approx $\sqrt{9.2}$

$$\sqrt{9.2} \approx 3 + \frac{1}{6}(9.2 - 9) = 3 + \frac{.2}{6} = \frac{18.2}{6} \text{ or } \boxed{\frac{91}{30}} \text{ or } 3.0\overline{3}$$

48 Given that $\frac{d}{dx} \sin x = \cos x$ use linearization to approximate $\sin(62°)$. Don't use a calculator. Leave in terms of π, squareroots, and fractions.

use $f(x) = \sin x$ and $a = 60° = \frac{\pi}{3}$

$f'(\frac{\pi}{3}) = \cos(\frac{\pi}{3}) = \frac{1}{2}$ $\quad f(\frac{\pi}{3}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$

$$l_{\frac{\pi}{3}}(x) = \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

$62° - 60° = 2° \cdot \frac{\pi}{180} = \frac{\pi}{90}$

$$\sin(62°) \approx \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{\pi}{90}\right)$$

$$= \boxed{\frac{\sqrt{3}}{2} + \frac{\pi}{180}} \text{ or } \frac{90\sqrt{3} + \pi}{180}$$

Extreme Values, Direction and the sign of the derivative, Convexity

49

(a) f has a local minimum of 2 at point −4

f has a local minimum of 2 at point 4

f has a local maximum of 6 at point 0

f has a local maximum of 8 at point 6

f has a global minimum of 2 at points −4 and 4

f has a global maximum of 8 at point 6

(b) f is increasing on $[-4,0]$, $[4,6]$

f is decreasing on $[-8,-4]$, $[0,4]$

(c) f' is negative on $(-8,-4)$, $(0,4)$

f' is positive on $(-4,0)$, $(4,6)$

f' is zero at point 0

(d)

x	−8	(−8,−4)	−4	(−4,0)	0	(0,4)	4	(4,6)	6	
$f(x)$	undef.	↘	min	↗	max	↘	min	↗	max	shape
$f'(x)$	undef.	−	undef	+	0	−	undef.	+	undef	sign
		↘		↘		↘		↗		direction
$f''(x)$	undef	−	undef	−	0	−	undef	+	undef	sign

50

(a) $f'(x) > 0$ on $(-\infty,-4)$, $(-2,-1)$, $(2,4)$, $(4,\infty)$

(b) $f'(x) < 0$ on $(-4,-2)$

(c) $f'(x) = 0$ on $(-1,1)$, $(1,2)$

(d) $f''(x) > 0$ on $(-4,-2)$, $(-2,-1)$

(e) $f''(x) < 0$ NonE

(f) $f''(x) = 0$ on $(-\infty,-4)$, $(-1,1)$, $(1,2)$, $(2,4)$, $(4,\infty)$