Chapter 3 Review Solutions

1) $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

2) a) $9x^2 - \dfrac{1}{\sqrt{1-x^2}}$

b) $\dfrac{4(5-3x^2) - (-6x)(4x-9)}{(5-3x^2)^2}$

c) $3 \cdot 2^t + 3t\,2^t \ln(2)$

d) $\dfrac{32w^7 + \frac{3}{w^2}}{4w^8 - \frac{3}{w}}$

e) $6s(4e^s + s^e)^{100} + 3s^2 \cdot 100(4e^s + s^e)^{99}(4e^s + es^{e-1}) = f'(s)$

$$f''(s) = 6(4e^s+s^e)^{100} + 6s \cdot 100(4e^s+s^e)^{99}(4e^s+es^{e-1})$$
$$+ \left[600s(4e^s+s^e)^{99} + 300s^2 \cdot 99(4e^s+s^e)^{98}(4e^s+es^{e-1})\right](4e^s+es^{e-1})$$
$$+ 300s^2(4e^s+e^s)^{99}(4e^s+e(e-1)s^{e-2}$$

f) $f'(v) = 0 - 0 + 3v^2 + 3^v \ln 3 + 0$

$f''(v) = 6v + 3^v(\ln 3)^2$

g) $9u^2(4u^8 - 9u + 4u\cos(u))^{12} + 3u^3 \cdot 12(4u^8 - 9u + 4u\cos(u))^{11} \cdot$

$(32u - 9 + 4\cos u - 4u\sin(u))$

h) $\sin z^{\cos z} = e^{\cos z \ln(\sin z)}$

$\Rightarrow e^{\cos z \ln(\sin z)}\left(-\sin z \ln(\sin z) + \dfrac{\cos z}{\sin z} \cdot \cos z\right)$

i) $f'(x) = \dfrac{3^x \ln 3 - 7}{(3^x - 7x)\ln 2}$

$f''(x) = \dfrac{3^x(\ln 3)^2(3^x-7x) - (3^x \ln 3 - 7)(3^x \ln 3 - 7)}{\ln 2 \cdot (3^x - 7x)^2}$

2j) $\sec^2(e^{t^4})[e^{t^4} \cdot 4t^3]$

3) $s^{3s} = e^{3s\ln s}$

$f'(s) = e^{3s\ln s}\left(3\ln s + \frac{3s}{s}\right) = s^{3s}(3\ln s + 3)$

$f'(1) = 1^{3(1)}(3\ln(1) + 3) = \boxed{3}$

4) $g'(t) = \frac{1}{1+(t^2+1)^2} \cdot 2t$

$g'(1) = \frac{1}{1+2^2} \cdot 2 = \boxed{\frac{2}{5}}$

5) $g'(t) = 12t^3$

$g'(2) = 12 \cdot 2^3 = \boxed{96}$

$g(2) = 48$

$\frac{d}{dt} g(t)\Big|_{t=2} = \frac{d}{dt} g(2) = \frac{d}{dt}(48) = \boxed{0}$

6) let $y = -3x$ $\quad x = -\frac{1}{3}y \quad \frac{1}{x} = \frac{-3}{y}$

$\lim_{y\to 0^-}\left((1+y)^{\frac{1}{y}}\right)^{-3} = e^{-3} = \boxed{\frac{1}{e^3}}$

7) let $y = \frac{2}{x} \Rightarrow x = \frac{2}{y}$

$\lim_{y\to 0}\left((1+y)^{\frac{1}{y}}\right)^2 = \boxed{e^2}$

8) $y' = 2x\,2^x + x^2 2^x \ln(2) = 0$

$2x\,\cancel{2^x} = -x^2 \cancel{2^x} \ln(2)$

$2 = -x\ln(2)$

$\boxed{x = -\frac{2}{\ln 2} \text{ or } x = 0}$

9) a) $f'(x) = \frac{1}{e^x + e^{2x}} \cdot e^x + 2e^{2x}$

$f(0) = \ln(1+1) = \ln 2$

$f'(0) = \frac{1+2}{1+1} = \frac{3}{2} = m$

$\boxed{l_0(x) = \ln 2 + \frac{3}{2}(x-0)}$

b) $f'(x) = \sec^2 x$

$f\left(\frac{\pi}{6}\right) = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$

$f'\left(\frac{\pi}{6}\right) = \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{4}{3} = m$

$\boxed{l_{\frac{\pi}{6}}(x) = \frac{1}{\sqrt{3}} + \frac{4}{3}\left(x - \frac{\pi}{6}\right)}$

10) $p'(t) = \frac{2t}{t^2-1}$ p is not differentiable at $p=1$ and $p=-1$

11) $\lim_{t\to 0} \frac{\sin(4t) \cdot 4}{3t \cdot 4} = \lim_{t\to 0} \frac{4}{3} \frac{\sin 4t}{(4t)} = \boxed{\frac{4}{3}}$

$\lim_{t\to 0} \frac{(\cos(2t) - 1)(\cos 2t + 1)}{5t(\cos(2t)+1)} = \lim_{t\to 0} \frac{\cos^2(2t) - 1}{5t(\cos(2t+1))}$

$= \lim_{t\to 0} \frac{-\sin^2(2t) \cdot 2}{2 \cdot 5t(\cos(2t)+1)} = \lim_{t\to 0} -\frac{2}{5} \frac{\sin 2t}{2t} \frac{\sin(2t)}{\cos(2t)+1} = -\frac{2}{5}(1) \cdot \frac{0}{1+1}$

$\boxed{= 0}$

12) $h = f \circ g$

$g(2)$ same as y of tan line

so $3(2) - 9 = -3$

$h'(2) = f'(g) \cdot g' = f'(g(2)) \cdot g'(2)$

$g'(2)$ is slope of tan line $= 3$

$= f'(-3) \cdot 3$

$f'(x) = e^{4x} \cdot 4$

$= 4e^{-12} \cdot 3 = \boxed{12e^{-12}}$

13) plug in $x = 1$ and set =

$a^2 - a = a + 3$

$a^2 - 2a - 3 = 0$

$(a-3)(a+1) = 0$

$a = 3$ or $a = -1$

check values for same derivative/slope

$f'(x) = \begin{cases} a^2 & x \le 1 \\ a + 6x & x > 1 \end{cases}$

$a = 3$: $a^2 = 9$, $a + 6(1) = 9$ same

$a = -1$: $a^2 = 1$, $a + 6(1) = 5$ not same

$\boxed{a = 3}$

14) a) $(fg)'(3) = f'(3)g(3) + f(3)g'(3) = (-1)(-4)+(9)(2) = \boxed{22}$

b) $\left(\frac{g}{fh}\right)' = \frac{g'fh - g(f'h+fh')}{(fh)^2}$ at 3 $\Rightarrow \frac{2\cdot 9\cdot 2-(-4)((-1)(2)+9(-5))}{(9\cdot 2)^2} = \boxed{-\frac{38}{81}}$

c) $(f\circ h)' = f'(h)\cdot h'$ at 3 $\Rightarrow f'(2)\cdot(-5) = 6(-5) = \boxed{-30}$

d) $(f^{-1})' = \frac{1}{f'(f^{-1}(9))} = \frac{1}{f'(3)} = \frac{1}{-1} = \boxed{-1}$

e) $3f'(3)+g'(3)-6h'(3) = 3(-1)+2-6(-5) = \boxed{29}$

f) $f'g+fg'-g'h-gh'$ at 3 $\Rightarrow (-1)(-4)+9(2)-2(2)-(-4)(-5) = \boxed{-2}$

g) $K'(f(3))\cdot f'(3) = K'(9)\cdot(-1) = 2(9)(-1) = \boxed{-18}$

15) a) $q'(p(0))\cdot p'(0) = \underset{\uparrow \text{DNE}}{q'(-1)}\ \frac{1}{2} \Rightarrow \boxed{\text{DNE}}$

b) $q'(p(2))\,p'(2) = q'(2)\cdot 2 = (-1)(2) = \boxed{-2}$

c) $p'(q(2))q'(2) = \underset{\uparrow \text{DNE}}{p'(1)}\cdot(-1) \Rightarrow \boxed{\text{DNE}}$

d) $p'q+pq' = p'(-2)q(-2)+p(-2)q'(-2) = (-2)(-1)+(1)(3) = \boxed{5}$

e) $\underset{\uparrow \text{DNE}}{p'(1)} - \overset{\swarrow \text{DNE}}{q'(1)} \Rightarrow \boxed{\text{DNE}}$

f) $\frac{p'(2)q(2)-p(2)q'(2)}{(q(2))^2} = \frac{(2)(1)-2(-1)}{(1)^2} = \boxed{4}$

16) $f'(3x^2)\cdot 6x$

17) $\frac{d}{dx}f(f^{-1}(x)) = \frac{d}{dx}(x) \Rightarrow f'(f^{-1}(x))\frac{d}{dx}f^{-1}(x) = 1$

$\Rightarrow \frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$

18) let $h = \frac{f}{g}$ prove $h' = \frac{f'g-fg'}{g^2}$

$h = fg^{-1}$ use product rule $f'g^{-1}+f(-g^{-2})g'$

$\frac{g}{g}\cdot\frac{f'}{g} - \frac{fg'}{g^2} \Rightarrow \frac{f'g-fg'}{g^2}$

19) $\lim_{h\to 0} \frac{f(3x+3h)-f(3x)}{h}\left(\frac{3}{3}\right)$

let $3h = h_1$ and $3x = x_1$

$3\lim_{h_1\to 0}\frac{f(x_1-h_1)-f(x_1)}{h_1}$

$= 3f'(x_1) = \boxed{3f'(3x)}$

20) $\dfrac{d}{dx}\, x f(x)$

$$= \lim_{h \to 0} \frac{(x+h) f(x+h) - x f(x)}{h}$$

$$= \lim_{h \to 0} \frac{x f(x+h) - h f(x+h) - x f(x)}{h}$$

$$= \lim_{h \to 0} \frac{h f(x+h)}{h} + \frac{x\,(f(x+h) - f(x))}{h}$$

$$= \lim_{h \to 0} f(x+h) + \lim_{h \to 0} x \left(\frac{f(x+h) - f(x)}{h} \right)$$

$$= f(x) + x f'(x)$$

21) $\cos x = \sin\left(x + \frac{\pi}{2}\right)$

$$\frac{d}{dx} \cos x = \frac{d}{dx} \sin\left(x + \frac{\pi}{2}\right)$$

$$= \cos\left(x + \frac{\pi}{2}\right)(1)$$

$$= -\sin x$$