Power Rule

1. $f(x) = x - x^3$

$= x^1 - x^3$

$= 1x^{1-1} - 3x^{3-1}$

$= x^0 - 3x^2$

$\boxed{f'(x) = 1 - 3x^2}$

2. $f(x) = \frac{4}{x^2} - \frac{x^2}{4}$

$= 4x^{-2} - \frac{1}{4}x^2$

$= 4(-2)x^{-2-1} - \frac{1}{4}(2)x^{2-1}$

$f'(x) = -8x^{-3} - \frac{1}{2}x$

$\boxed{f'(x) = -\frac{8}{x^3} - \frac{x}{2}}$

3. $h(x) = \frac{3}{\sqrt{x}}$

$= 3x^{-1/2}$

$= 3\left(-\frac{1}{2}\right)x^{-1/2-1} = -\frac{3}{2}x^{-3/2}$

$\boxed{h'(x) = -\frac{3}{2}x^{-3/2}}$

4. $f(x) = x^2 - e^2$

$= 2x^{2-1} - 0$

e^2 is constant

$\boxed{f'(x) = 2x}$

5. $g(x) = \sqrt{\sqrt{x}}$

$= \left(x^{\frac{1}{2}}\right)^{\frac{1}{2}} = x^{\frac{1}{4}}$

$\frac{1}{4}x^{\frac{1}{4}-1}$

$\boxed{g'(x) = \frac{1}{4}x^{-3/4}}$

6. $f(x) = \frac{x^2 - 1}{x}$

$= x - \frac{1}{x}$

$= x - x^{-1}$

$f'(x) = 1 - (-1)x^{-1-1}$

$f'(x) = 1 + x^{-2} = \boxed{1 + \frac{1}{x^2}}$

7. $f(x) = \frac{7x + 3x^2}{5\sqrt{x}}$

$= \frac{7}{5}x^{\frac{1}{2}} + \frac{3}{5}x^{\frac{3}{2}}$

$f'(x) = \frac{7}{5} \cdot \frac{1}{2}x^{\frac{1}{2}-1} + \frac{3}{5} \cdot \frac{3}{2}x^{\frac{3}{2}-1}$

$= \frac{7}{10}x^{-1/2} + \frac{9}{10}x^{\frac{1}{2}}$

$\boxed{f'(x) = \frac{7}{10\sqrt{x}} + \frac{9\sqrt{x}}{10}}$

Chain Rule

8. $f(x) = (x^2 - 1)^{10}$

$f'(x) = 10(x^2-1)^9 \frac{d}{dx}(x^2-1)$

$\boxed{f'(x) = 10(x^2-1)^9(2x)}$

9. $f(x) = \sqrt{1+\sqrt{1+2x}}$

$= \left(1+(1+2x)^{\frac{1}{2}}\right)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}\left(1+(1+2x)^{\frac{1}{2}}\right)^{-\frac{1}{2}} \frac{d}{dx}\left(1+(1+2x)^{\frac{1}{2}}\right)$

$f'(x) = \frac{1}{2}\left(1+(1+2x)^{\frac{1}{2}}\right)^{-\frac{1}{2}}\left(\frac{1}{2}(1+2x)^{-1/2}\right)\frac{d}{dx}(1+2x)$

$\boxed{f'(x) = \frac{1}{2}\left(1+(1+2x)^{\frac{1}{2}}\right)^{-\frac{1}{2}}\left(\frac{1}{2}(1+2x)^{-1/2}\right)(2)}$

10. $g(x) = (3x^2 + 3x - 6)^{-8}$

$g'(x) = -8(3x^2+3x-6)^{-8-1} \frac{d}{dx}(3x^2+3x-6)$

$\boxed{g'(x) = -8(3x^2+3x-6)^{-9}(6x+3)}$

11. $f(x) = \sqrt[4]{9-x}$

$= (9-x)^{\frac{1}{4}}$

$f'(x) = \frac{1}{4}(9-x)^{\frac{1}{4}-1} \frac{d}{dx}(9-x)$

$\boxed{f'(x) = \frac{1}{4}(9-x)^{-\frac{3}{4}}(-1)}$

Product and Quotient Rule

12. $f(x) = (x+1)(x^2-3)$ (Try this one in two different ways.)

① product

$\boxed{\frac{d}{dx}(x+1)}(x^2-3)+(x+1)\boxed{\frac{d}{dx}(x^2-3)}$

$(x^2-3)+(x+1)(2x)$

$3x^2+2x-3$

② multiply

$f(x) = x^3+x^2-3x-3$

$f'(x) = 3x^2+2x-3$

13. $g(x) = \dfrac{3x^2+5x}{\sqrt{x}}$

$$g'(x) = \frac{\overset{f'}{(6x+5)}\overset{h}{\sqrt{x}} - \overset{f}{(3x^2+5x)}\overset{h'}{\left(\tfrac{1}{2}x^{-1/2}\right)}}{\underset{h^2}{x}}$$

14. $f(x) = \overset{g}{x}\overset{h}{\sqrt{3x^2-x}}$

$$f'(x) = \overset{g'}{1}\overset{h}{\sqrt{3x^2-x}} + \overset{g}{x}\underbrace{\left(\tfrac{1}{2}(3x^2-x)^{-1/2}(6x-1)\right)}_{h' \text{ with chain rule}}$$

15. $f(x) = \dfrac{\overbrace{(5x^2-3)(x^2-2)}^{g}}{\underset{h}{x^2+2}}$

$$f'(x) = \frac{\overbrace{\left[\overset{g_1'}{10x}\overset{g_2}{(x^2-2)}+\overset{g_1}{(5x^2-3)}\overset{g_2'}{(2x)}\right]}^{g'}\overset{h}{(x^2+2)}-\overset{g}{(5x^2-3)(x^2-2)}\overset{h'}{(2x)}}{(x^2+2)^2}$$

16. $g(x) = \dfrac{\overset{f}{x}}{\underset{h}{x+\frac{17}{x}}}$

$$g'(x) = \frac{\overset{f'}{1}\overset{h}{\left(x+\frac{17}{x}\right)} - \overset{f}{x}\overset{h'}{\left(1-\frac{17}{x^2}\right)}}{\left(x+\frac{17}{x}\right)^2}$$

17. $h(x) = \overset{f}{(\sqrt{x}-4)^3}\overset{g}{(\sqrt{x}+4)^5}$

$$h'(x) = \underbrace{3(\sqrt{x}-4)^2\left(\frac{1}{2\sqrt{x}}\right)}_{f' \text{ with chain rule}}\underbrace{(\sqrt{x}+4)^5}_{g} + \underbrace{(\sqrt{x}-4)^3}_{f}\cdot\underbrace{5(\sqrt{x}+4)^4\left(\frac{1}{2\sqrt{x}}\right)}_{g' \text{ with chain rule}}$$

All Mixed Up: Power, Product, Quotient, Chain Rules

18. Find the first derivative of the following functions:

(a) $f(t) = 3t^2 + 2t$

$f'(t) = 6t + 2$

(b) $g(w) = \dfrac{w^3}{(w+3)^5}$

$g'(w) = \dfrac{(w+3)^5 \cdot 3w^2 - w^3 \cdot 5(w+3)^4}{(w+3)^{10}}$

(c) $h(s) = (s^{-2})^3 = s^{-6}$

$h'(s) = -6s^{-7}$

(d) $f(x) = 5\sqrt{x}$ at 4

$f'(x) = \dfrac{5}{2\sqrt{x}}$

So $f'(4) = \dfrac{5}{2\sqrt{4}} = \dfrac{5}{4}$

(e) $g(x) = \sqrt[3]{\sqrt[5]{\sqrt{x}}} = \left(\left(x^{1/2}\right)^{1/5}\right)^{1/3} = x^{1/30}$

$g'(x) = \dfrac{1}{30} x^{-29/30}$

(f) $f(x) = \pi^2$ constant

$f'(x) = 0$

(g) $m(t) = \sqrt{t^2 - 5t} = (t^2 - 5t)^{1/2}$

$m'(t) = \frac{1}{2}(t^2 - 5t)^{-1/2}(2t - 5)$

$= \dfrac{2t-5}{2\sqrt{t^2-5t}}$

(h) $g(y) = \sqrt{1 + \sqrt{1 + \sqrt{y}}} = \left(1 + \left(1 + y^{1/2}\right)^{1/2}\right)^{1/2}$

$g'(y) = \frac{1}{2}\left(1 + \left(1 + y^{1/2}\right)^{1/2}\right)^{-1/2} \cdot \left(\frac{1}{2}\left(1 + y^{1/2}\right)^{-1/2}\right) \cdot \frac{1}{2} y^{-1/2}$

(i) $h(s) = (s+1)^5\sqrt{s-1}$

$h'(s) = 5(s+1)^4\sqrt{s-1} + (s+1)^5 \dfrac{1}{2\sqrt{s-1}}$

product rule

(j) $f(x) = \dfrac{2x-1}{\sqrt{x+1}}$

$f'(x) = \dfrac{\sqrt{x+1}\,(2) - (2x-1)\dfrac{1}{2\sqrt{x+1}}}{x+1}$

(k) $f(x) = \dfrac{(x+2)^2(3x-4x^5)^{100}}{(8-x)^7}$

$$f'(x) = \frac{\left[(8-x)^7\left[2(x+2)(3x-4x^5)^{100} + (x+2)^2 \cdot 100(3x-4x^5)^{99}(3-20x^4)\right] - (x+2)^2(3x-4x^5)^{100} \cdot 7(8-x)^6(-1)\right]}{(8-x)^{14}}$$

Derivatives of Exponential Functions

Recall: $\frac{d}{dx}e^x = e^x$, $\frac{d}{dx}e^{g(x)} = e^{g(x)} \cdot g'(x)$

19. Find the first derivative of the following functions:

$a>0$: $\frac{d}{dx}a^x = a^x \ln a$, $\frac{d}{dx}a^{g(x)} = a^{g(x)}\ln(a)g'(x)$

(a) $f(t) = e^{3t}$

$f'(t) = e^{3t} \cdot \frac{d}{dt} 3t = e^{3t} \cdot 3 = \boxed{3e^{3t}}$

(b) $g(z) = \left(\frac{2}{3}\right)^{3z-z^2}$

$g'(z) = \left(\frac{2}{3}\right)^{3z-z^2} \cdot \ln\left(\frac{2}{3}\right) \cdot \frac{d}{dz}(3z - z^2) = \boxed{\left(\frac{2}{3}\right)^{3z-z^2} \cdot \ln\left(\frac{2}{3}\right) \cdot (3-2z)}$

(c) $h(k) = 7e^{-5} - 7e^{-5k} + k^2 \ln(e^4)$

constant

$h'(k) = -7e^{-5k} \cdot \frac{d}{dk}(-5k) + 2k \cdot \underbrace{\ln(e^4)}_{=4} = -7e^{-5k}(-5) + 2k(4)$

$= \boxed{35e^{-5k} + 8k}$

(d) $i(r) = 2^{4\sqrt{r}}$

$i'(r) = 2^{4\sqrt{r}} \cdot \ln(2) \cdot \frac{d}{dr} 4\sqrt{r} = 2^{4\sqrt{r}} \cdot \ln(2) \cdot 4 \cdot \frac{1}{2\sqrt{r}} = 2^{4\sqrt{r}} \cdot \ln(2) \cdot \frac{2}{\sqrt{r}}$

$= \boxed{2^{4\sqrt{r}+1} \cdot \frac{\ln(2)}{\sqrt{r}}}$

(e) $A(t) = Pe^{rt}$ where P, r are constants

$A'(t) = \frac{d}{dt}Pe^{rt} = P\frac{d}{dt}e^{rt} = P \cdot e^{rt} \cdot \frac{d}{dt} rt = Pe^{rt} \cdot r = \boxed{Pre^{rt}}$

Note: $A'(t) = rA(t)$

i.e. $y' = r \cdot y$

Derivatives of Logarithmic Functions

Recall: $\frac{d}{dx}\ln x = \frac{1}{x} \xrightarrow[\text{rule application}]{\text{chain}} \frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$

20. Find the first derivative of each function. $a>0, a\neq 1: \frac{d}{dx}\log_a x = \frac{1}{x\ln a}$

(a) $l(t) = \ln(x^2 - 1)$

$$l'(t) = \frac{\frac{d}{dx}(x^2-1)}{x^2-1} = \boxed{\frac{2x}{x^2-1}}$$

(b) $h(x) = \ln(x^x)$

logarithm rules!

$$h'(x) = \frac{d}{dx}\ln(x^x) = \frac{d}{dx}x\ln x = 1\ln x + x\cdot\frac{1}{x} = \boxed{\ln x + 1}$$

product rule

(c) $t(y) = y\ln\frac{1}{y}$

$f(y) = 1/y$ $\quad \frac{f'(y)}{f(y)}$

$$t'(y) = \frac{d}{dy}y\ln\frac{1}{y} = 1\cdot\ln\left(\frac{1}{y}\right) + y\cdot\frac{\frac{d}{dy}(1/y)}{1/y} = \ln\left(\frac{1}{y}\right) + y\,\frac{-1/y^2}{1/y}$$

$$= \boxed{\ln\left(\frac{1}{y}\right) + -1}$$

product rule

(d) $j(x) = \ln\left(\frac{(4x-1)^8(3x^2+14)^7}{\sqrt{x^2-4}}\right)$ Going to make this easier by doing some log rule applications

$$j(x) = \ln(4x-1)^8 + \ln(3x^2+14)^7 - \ln(x^2-4)^{1/2} = 8\ln(4x-1) + 7\ln(3x^2+14) - \tfrac{1}{2}\ln(x^2-4)$$

Now differentiation is much easier.

$$j'(x) = \frac{d}{dx}\left(8\ln(4x-1) + 7\ln(3x^2+14) - \tfrac{1}{2}\ln(x^2-4)\right) = 8\left(\frac{\frac{d}{dx}(4x-1)}{4x-1}\right) + 7\left(\frac{\frac{d}{dx}(3x^2+14)}{(3x^2+14)}\right) - \frac{1}{2}\left(\frac{\frac{d}{dx}(x^2-4)}{x^2-4}\right)$$

$$= 8\left(\frac{4}{4x-1}\right) + 7\left(\frac{6x}{3x^2+14}\right) - \frac{1}{2}\left(\frac{2x}{x^2-4}\right)$$

$$= \boxed{\frac{32}{4x-1} + \frac{42x}{3x^2+14} - \frac{x}{x^2-4}}$$

(e) $k(s) = \log_2((5s^8 - 11)^3)$

$$k(s) = \frac{\ln(5s^8-11)^3}{\ln(2)} = \frac{3}{\ln(2)}\cdot\ln(5s^8-11)$$

"change of base"

Therefore $k'(s) = \frac{d}{ds}\frac{3}{\ln(2)}\ln(5s^8-11) = \frac{3}{\ln(2)}\cdot\frac{\frac{d}{ds}(5s^8-11)}{5s^8-11}$

$$= \frac{3}{\ln(2)}\,\frac{40s^7}{5s^8-11} = \boxed{\frac{120s^7}{\ln(2)(5s^8-11)}}$$

Recall $(\cos)' = -\sin$, $(\sin)' = \cos$, $\tan' = \sec^2$

$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$, $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$(\arctan x)' = \frac{1}{1+x^2}$

Derivatives of Trig / Inverse Trig / Inverse Functions

21. Find the first derivative of each function

(a) $a(s) = 2\sin^2(s) + 2\cos^2(s)$
Do this one two ways.

① Note $a(s) = 2(\sin^2(s) + \cos^2(s))$
$= 2(1)$ ← Pythagorean identity.
$= 2$

So $a'(s) = \boxed{0}$

② $a'(s) = \frac{d}{ds}(2\sin^2(s) + 2\cos^2(s))$ chain rules

$= 2 \cdot 2\sin(s)\cos(s) + 2 \cdot 2\cos(s)(-\sin(s))$

$= 4\sin(s)\cos(s) - 4\cos(s)\sin(s)$

$= \boxed{0}$

same!

(b) $d(v) = \arccos(\cos(v))$
Do this one two ways.

① Recall: $d(v) = \arccos(\cos(v))$
$= v$
since $\cos$, $\arccos$ are inverse functions
Thus $d'(v) = \boxed{1}$

SAME!

② Using chainrule and an identity: $1 - \cos^2 v = \sin^2 v$

$d'(v) = \arccos'(\cos(v)) \cdot \cos'(v)$

$= -\frac{1}{\sqrt{1-\cos^2 v}} \cdot (-\sin v)$

$= -\frac{1}{\sqrt{\sin^2 v}} \cdot (-\sin v)$

since $\sin(x) \geq 0$ on the restricted domain for $\cos(x)$ of $[0, \pi]$ then $\sqrt{\sin^2 x} = \sin x$

$= -\frac{1}{\sin x} \cdot (-\sin x)$

$= \boxed{1}$

(c) $b(t) = 4\ln(5\cos(t))$

chainrule

$b'(t) = 4\frac{d}{dt}\ln(5\cos(t)) = \frac{4 \cdot \frac{d}{dt} 5\cos(t)}{5\cos(t)} = \frac{4 \cdot 5(-\sin(t))}{5\cos(t)} = \boxed{-4\tan(t)}$

Note: $b(t) = 4\ln(5) + 4\ln(\cos(t))$ so $b'(t) = 0 + 4\frac{\frac{d}{dt}\cos t}{\sin t} = 4\frac{-\sin t}{\cos t} = \boxed{-4\tan t}$
(as another approach)

(d) $c(u) = \cos(\sin(u))$

Using chainrule: $c'(u) = \cos'(\sin(u)) \cdot \sin'(u) = \boxed{-\sin(\sin(u)) \cdot \cos(u)}$

which is NOT $-\sin^2(u)\cos(u)$

(e) $f(w) = \tan(w^2 + 1)$

$f'(w) = \tan'(w^2+1)\frac{d}{dw}(w^2+1) = \boxed{\sec^2(w^2+1)(2w)}$

chain rule (again...)

(f) $g(v) = \arcsin(\cos(v)) + \cos(\arcsin(v))$ and simplify your result

$$g'(v) = \arcsin'(\cos(v))\cdot\cos'(v) + \cos'(\arcsin(v))\cdot\arcsin'(v)$$
$$= \frac{1}{\sqrt{1-\cos^2 v}}(-\sin v) + \underbrace{-\sin(\arcsin(v))}\cdot\frac{1}{\sqrt{1-v^2}}$$
$$= \frac{1}{\sqrt{\sin^2 v}}(-\sin v) + -v\cdot\frac{1}{\sqrt{1-v^2}} = \boxed{-1-\frac{v}{\sqrt{1-v^2}}}$$

Note also: $\cos(\arcsin(v))$
$$= \sqrt{1-\sin^2(\arcsin(v))}$$
$$= \sqrt{1-v^2}$$
& $\frac{d}{dv}\sqrt{1-v^2} = \frac{-v}{\sqrt{1-v^2}}$

(g) $h(y) = y^2 \arctan(4y)$

product rule & chain rule

$$h'(y) = 2y\arctan(4y) + y^2\cdot\frac{1}{1+(4y)^2}\cdot\frac{d}{dy}4y$$
$$= \boxed{2y\arctan(4y) + \frac{4y^2}{1+(4y)^2}}$$

(h) $i(z) = \sec^7(2z)$

chain rule

$$i'(z) = 7\sec^6(2z)\cdot\frac{d}{dz}\sec(2z) = 7\sec^6(2z)\cdot\frac{d}{dz}(\cos(2z))^{-1}$$ chain rule x2
$$= 7\sec^6(2z)\cdot(-1)(\cos(2z))^{-2}\cdot(-\sin(2z))\cdot 2$$
$$= \boxed{14\sin(2z)\cdot\sec^8(2z)}$$
since $\cos(2z)^{-2} = \sec^2(2z)$

22. If $f(2) = \cancel{7}\,4$ and $f'(\cancel{7}\,2) = \cancel{4}\,7$ determine the derivative of f^{-1} at 4.

Recall that $\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$ and if we want $(f^{-1})'$ at 4 then we need $f'(f^{-1}(4)) = f'(2) = 7$ thus the derivative of f^{-1} at 4 is $1/7$

23. If $f(x) = \dfrac{2x-1}{3x+4}$ determine $\dfrac{d}{dx}f^{-1}(x)$

(a) using the result relating f' and $(f^{-1})'$ obtained in class.

(b) by determining f^{-1} and then differentiating.

a) $f'(x) = \frac{(3x+4)2-(2x-1)(3)}{(3x+4)^2} = \frac{11}{(3x+4)^2}$

quotient rule

Then:

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{11/(3f^{-1}(x)+4)^2}$$
using f^{-1} found in part (b)
$$= \frac{\left(3\left(\frac{-1-4x}{3x-2}\right)+4\right)^2}{11} = \frac{\left(\frac{-3-12x}{3x-2}+\frac{12x-8}{3x-2}\right)^2}{11} = \frac{\left(\frac{-11}{3x-2}\right)^2}{11} = \frac{11}{(3x-2)^2}$$

b) $x = f(f^{-1}(x)) = \frac{2f^{-1}(x)-1}{3f^{-1}(x)+4}$

⋮ algebra to solve for $f^{-1}(x)$

$$f^{-1}(x) = \frac{-1-4x}{3x-2}$$

Then $\frac{d}{dx}f^{-1}(x) = \frac{(3x-2)(-4)-(-1-4x)(3)}{(3x-2)^2}$
$$= \frac{11}{(3x-2)^2}$$

SAME!

All Mixed Up: Derivatives of Exponentials, Logarithms, Trig, Misc. Functions

24. Find the first derivative of the following functions:

(a) $f(t) = 3e^{4t}$

$f'(t) = 12e^{4t}$

(b) $g(w) = \frac{e^3}{(3)^w} = \underbrace{e^3}_{\text{constant}} \cdot 3^{-w}$

$g'(w) = -e^3 3^{-w} \cdot \ln(3)$

(c) $h(s) = e^{2s}\ln(2s)$ at $1/2$

(product rule) $h'(s) = (2e^{2s})\ln(2s) + e^{2s} \cdot \frac{1}{s}$

$h'(\frac{1}{2}) = 2e^{2(1/2)} \cdot \ln(2 \cdot \frac{1}{2}) + e^{2(1/2)} \cdot \frac{1}{1/2} = 0 + 2e = \boxed{2e}$

(d) $f(x) = 5\sqrt{\log_3(x)} = 5\left(\frac{\ln x}{\ln 3}\right)^{1/2}$

$f'(x) = \frac{5}{2} \cdot \left(\frac{\ln x}{\ln 3}\right)^{-1/2} \cdot \frac{1}{x \ln 3}$

or $= \frac{5}{2\sqrt{\log_3(x)}} \cdot \frac{1}{x \ln 3}$

(e) $g(x) = x^2 e^{x^2}$

(product & chain rules) $g'(x) = (2x)e^{x^2} + x^2(e^{x^2} \cdot 2x)$

$= 2xe^{x^2} + 2x^3 e^{x^2}$

(f) $f(x) = x^e$ (power function)

$f'(x) = ex^{e-1}$

(g) $f(x) = (\pi e)^2$

$f'(x) = 0$; $(\pi e)^2$ is a constant

(h) $m(t) = \tan(3t)$

$m'(t) = 3\sec^2(3t)$ (chainrule)

(i) $g(y) = y\cos(\ln y))$ ↓ product & chain rules

$g'(y) = 1\cos(\ln(y)) + y\cos'(\ln(y)) \cdot \ln'(y)$

$= \cos(\ln(y)) + y(-\sin(\ln(y))) \cdot \frac{1}{y}$

$= \cos(\ln(y)) + -\sin(\ln(y))$

(j) $h(s) = s\sin s$

$h'(s) = 1\sin(s) + s\cos(s)$

$= \sin(s) + s\cos(s)$

(k) $f(x) = \frac{x}{\sin x}$

$f'(x) = \frac{1\sin(x) - x\cos x}{\sin^2 x} = \frac{\sin x - x\cos x}{\sin^2 x}$

quotient rule

(l) $f(x) = \frac{(x+2)^2(e)^{100+x^3}}{\sin^7(x)}$

$f'(x) = \frac{\sin^7 x \frac{d}{dx}\left[(x+2)^2 e^{100+x^3}\right] - (x+2)^2 e^{100+x^3} \cdot \frac{d}{dx}\sin^7(x)}{(\sin^7(x))^2}$

$= \frac{\sin^7 x\left(2(x+2)e^{100+x^3} + (x+2)^2 \cdot 3x^2 e^{100+x^3}\right) - (x+2)^2 e^{100+x^3}(7\sin^6 x \cdot \cos x)}{\sin^{14} x}$

Some "log trick" problems

25. Using the "log trick" show that $\frac{d}{dx}a^x = a^x \ln(a)$ Recall $a^x = e^{\ln a^x}$ & $\frac{d}{dx} e^{g(x)} = e^{g(x)} \cdot g'(x)$

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{\ln a^x} = \frac{d}{dx} e^{x \ln a} = e^{x\ln a} \cdot \frac{d}{dx} x \ln a = a^x \cdot \ln a$$

26. Use the "log trick" to show that $(x^x)' = x^x(\ln(x) + 1)$ First note x^x has domain $x>0$

$$\frac{d}{dx} x^x = \frac{d}{dx} e^{\ln x^x} = \frac{d}{dx} e^{x\ln x} = e^{x \ln x} \cdot \frac{d}{dx} x \ln x = x^x \cdot \left(\ln x + x \frac{1}{x}\right) = x^x(\ln x + 1)$$

product rule

27. Use the "log trick" and the previous problem to determine $\frac{d}{dx}x^{x^x}$.

$$\frac{d}{dx} x^{x^x} = \frac{d}{dx} e^{\ln(x^{x^x})} = \frac{d}{dx} e^{x^x \ln x} = e^{x^x \ln x} \cdot \frac{d}{dx} x^x \ln x$$

product rule

$$= x^{x^x} \cdot \left(\left(\frac{d}{dx} x^x\right) \ln x + x^x \frac{d}{dx} \ln x \right)$$

previous problem

$$= x^{x^x} \cdot \left(x^x(\ln x + 1)\ln x + x^x \cdot \frac{1}{x} \right)$$

28. If $g(d) = ab^2 + 3c^3d + 5b^2c^2d^2$, then what is $g''(d)$?

$g'(d) = 0 + 3c + 10b^2c^2d = 3c + 10b^2c^2d$

$g''(d) = 0 + 10b^2c^2 = 10b^2 + c^2$

29. If $\frac{dy}{dx} = 5$ and $\frac{dx}{dt} = -2$ then what is $\frac{dy}{dt}$?

$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx}$, $\frac{\frac{dy}{dt}}{-2} = 5$, $\frac{dy}{dt} = -2(5) = -10$

30. A ball is thrown into the air and its height h (in meters) after t seconds is given by the function $h(t) = 10 + 20t - 5t^2$. When the ball reaches its maximum height, its velocity will be zero.

(a) At what time will the ball reach its maximum height?

velocity of zero

$h'(t) = 0 + 20 - 10t$

$20 - 10t = 0$

$\frac{20}{10} = \frac{10t}{10}$

$\boxed{t = 2 \text{ seconds}}$

(b) What is the maximum height of the ball?

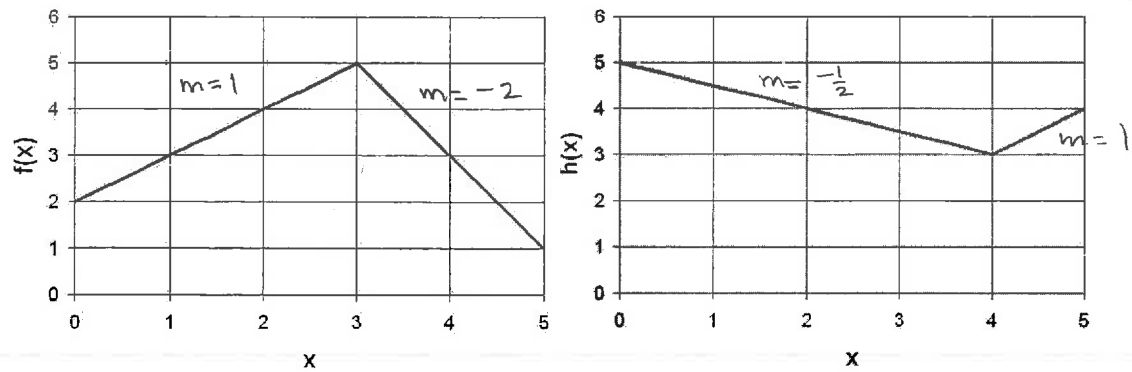
What is h when t=2?

$h(2) = 10 + 20(2) - 5(2)^2$

$= 10 + 40 - 20$

$= 30$ meters

31. Given the graphs of $f(x)$ and $h(x)$.



(a) The function $g = 10fh$. What is $g'(2)$?

$$g'(2) = 10f(2)h'(2) + 10h(2)f'(2)$$
$$= 10(4)(-\tfrac{1}{2}) + 10(4)(1)$$
$$= -20 + 40 = \boxed{20}$$

(b) The function $g = 10f(h)$. What is $g'(2)$?

$g = 10 f \circ h$ (composition, not multiplication)

$$g'(2) = 10 f'(h(2)) \cdot h'(2) = 10 f'(4) \cdot (-\tfrac{1}{2}) = 10(-2)(-\tfrac{1}{2}) = \boxed{10}$$

(c) The function $g = 10\frac{f}{h}$. What is $g'(2)$?

$$g'(2) = 10\left[\frac{h(2)f'(2) - f(2)h'(2)}{(h(2))^2}\right] = 10\left(\frac{(4)(1) - (4)(-\frac{1}{2})}{4^2}\right)$$
$$= 10\left(\frac{4+2}{16}\right) = 10\left(\frac{6}{16}\right) = \frac{60}{16} = \boxed{\frac{15}{4}}$$

32. What is the line tangent to $f(x) = x^3$ at 2?

$$f'(x) = 3x^2$$
$$m = f'(2) = 3(2)^2 = 12$$
$$y_1 = f(2) = 2^3 = 8$$

$(\underset{x_1}{2}, \underset{y_1}{8})$, $m = 12$

$$y - y_1 = m(x - x_1)$$
$$\boxed{y - 8 = 12(x - 2)}$$
$$y - 8 = 12x - 24$$
$$\boxed{y = 12x - 16}$$

33. Find the derivative in $f(x) = \dfrac{x}{\sqrt{x}}$ in three ways. i) using algebra and the power rule, ii) the product rule and iii) the quotient rule. Carry through algebra to show that these are all equal.

a) $f(x) = \frac{x}{\sqrt{x}} = x^1 x^{-\frac{1}{2}} = x^{\frac{1}{2}}$, $f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$

b) $f(x) = \frac{x}{\sqrt{x}} = x^1 x^{-\frac{1}{2}}$, $f'(x) = x\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) + x^{-\frac{1}{2}}(1)$

$= -\frac{1}{2}x^{-\frac{1}{2}} + x^{-\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}}$ ✓

c) $f(x) = \frac{x}{x^{\frac{1}{2}}}$, $f'(x) = \dfrac{x^{\frac{1}{2}}(1) - x\left(\frac{1}{2}x^{-\frac{1}{2}}\right)}{\left(x^{\frac{1}{2}}\right)^2} = \dfrac{x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}}}{x}$

$= \dfrac{\frac{1}{2}x^{\frac{1}{2}}}{x} = \frac{1}{2}x^{-\frac{1}{2}}$ ✓

34. Let $f(3) = 2$, $f'(3) = 4$, $g(3) = 1$, $g'(3) = 3$ and $f'(1) = 5$.

(a) If $h(x) = f(x)g(x)$, what is $h'(3)$?

$h'(x) = f(x)g'(x) + g(x)f'(x)$

$h'(3) = f(3)g'(3) + g(3)f'(3) = (2)(3) + (1)(4) = 6 + 4 = \boxed{10}$

(b) If $h(x) = \dfrac{f(x)}{g(x)}$, what is $h'(3)$?

$h'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

$h'(3) = \dfrac{g(3)f'(3) - f(3)g'(3)}{(g(3))^2} = \dfrac{(1)(4) - (2)(3)}{(1)^2} = \dfrac{4-6}{1} = \boxed{-2}$

(c) If $h(x) = f \circ g(x)$, what is $h'(3)$?

$h'(x) = f'(g(x))g'(x)$

$h'(3) = f'(g(3))g'(3) = f'(1)\cdot(3) = (5)(3) = \boxed{15}$

35. A function has a local minimum at $x = -1$ and $x = 3$ and a local max at $x = 2$. What is a possible function for $f'(x)$?

f	↘	↗	↘	↗
f'	$-$	$+$	$-$	$+$
	-1	2	3	

$f'(x) = (x+1)(x-3)(x-2)$

36. If $u = ve^w + xy^v$, then what is $\frac{du}{dv}$?

$$\frac{du}{dv} = e^w + xy^v \ln y$$

37. Use the product rule to show that the derivative of $\tan(x)$ is $\sec^2(x)$.

$$y = \tan x = \frac{\sin x}{\cos x} = \sin x \sec x$$

$$\frac{dy}{dx} = \sin x (\sec x \tan x) + \sec x (\cos x)$$

$$= \sin x \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) + 1$$

$$= \frac{\sin^2 x}{\cos^2 x} + 1$$

$$= \tan^2 x + 1$$

$$= \sec^2 x$$

38. For what value of x is $\frac{d}{dx}e^x = 1$?

$$\frac{d}{dx}(e^x) = e^x$$

$e^x = 1$ when $x = 0$

39. What is the line tangent to $f(x) = 2e^x$ at 1?

$$f'(x) = 2e^x$$

$$m = f'(1) = 2e^1 = 2e$$

$$y_1 = f(1) = 2e^1 = 2e$$

$\underset{x_1}{(1}, \underset{y_1}{2e)}$, $m = 2e$

$$y - y_1 = m(x - x_1)$$

$$\boxed{y - 2e = 2e(x-1)}$$

$$y = 2ex - 2e + 2e$$

$$\boxed{y = 2ex}$$

40. If $\ln(x) - y = 0$, find $\frac{dx}{dy}$.

$$\ln x = y$$

$$x = e^y$$

$$\frac{dx}{dy} = e^y$$

41. Let $f(x) = e^{x^2} \cos(2x)\sqrt{3x+1}$, find $f'(x)$.

$$f'(x) = e^{x^2}\left(\cos(2x)\left(\tfrac{1}{2}(3x+1)^{-\frac{1}{2}} \cdot 3 + \sqrt{3x+1}\,(-2\sin(2x))\right)\right)$$
$$+ \left(\cos(2x)\sqrt{3x+1}\right)\left(e^{x^2}(2x)\right)$$

42. Let $f(x) = \dfrac{x^3}{3} + x^2 - 3x$ for all $x \in \mathbf{R}$.

(a) For what values (there are two of them) is $f'(x) = 0$.

$f'(x) = x^2 + 2x - 3$

$x^2 + 2x - 3 = 0$ $\qquad x = -3, 1$

$(x+3)(x-1) = 0$

(b) List the intervals where f is increasing. Don't use a graph.

f	$\nearrow$		$\searrow$		$\nearrow$
f'	$+$	-3	$-$	1	$+$

$(-\infty, -3]$, $[1, \infty)$

(c) List the intervals where f is decreasing. Don't use a graph.

$[-3, 1]$

(d) Where does f have a local maximum?

$x = -3$

(e) What is the local minimum value of f?

$$f(-3) = \frac{(-3)^3}{3} + (-3)^2 - 3(-3)$$
$$= -9 + 9 - 9 = -9$$