L'Hospital's Rule

① $\lim\limits_{x\to 0} \dfrac{x}{e^{4x}-1} = \lim\limits_{x\to 0} \dfrac{1}{4e^{4x}} = \dfrac{1}{4} \lim\limits_{x\to 0} \dfrac{1}{e^{4x}} = \boxed{\dfrac{1}{4}}$

"$\dfrac{0}{0}$"

② $\lim\limits_{x\to \infty} \dfrac{x}{e^{4x}-1} = \lim\limits_{x\to \infty} \dfrac{1}{4e^{4x}} = \dfrac{1}{4} \lim\limits_{x\to \infty} \dfrac{1}{e^{4x}} = \dfrac{1}{4}(0) = \boxed{0}$

"$\dfrac{?}{\infty}$"

③ $\lim\limits_{x\to 2} \dfrac{x}{e^{4x}-1} = \dfrac{2}{e^{4(2)}-1} = \boxed{\dfrac{2}{e^{8}-1}}$

NOT INDETERMINATE FORM
DIRECT SUBSTITUTION

④ $\lim\limits_{x\to 0} \dfrac{\sin(3x)}{\sin(5x)} = \lim\limits_{x\to 0} \dfrac{3\cos(3x)}{5\cos(5x)} = \dfrac{3}{5} \lim\limits_{x\to 0} \dfrac{\cos(3x)}{\cos(5x)} = \dfrac{3}{5}(1) = \boxed{\dfrac{3}{5}}$

"$\dfrac{0}{0}$"

⑤ $\lim\limits_{x\to 2} \dfrac{2e^{x-2}-x}{x^2-4} = \lim\limits_{x\to 2} \dfrac{2e^{x-2}-1}{2x} = \boxed{\dfrac{1}{4}}$

"$\dfrac{0}{0}$"

⑥ $\lim\limits_{x\to \infty} x^2 e^{-x^2} = \lim\limits_{x\to \infty} \dfrac{x^2}{e^{x^2}} = \lim\limits_{x\to \infty} \dfrac{2x}{2x\,e^{x^2}} = \lim\limits_{x\to \infty} \dfrac{1}{e^{x^2}} = \boxed{0}$

"$0 \cdot \infty$"

⑦ $\lim\limits_{x\to 0} \dfrac{4x^3}{e^x} = \dfrac{4(0)^3}{e^0} = \boxed{0}$

NOT INDETERMINATE FORM
DIRECT SUBSTITUTION

(8) $\lim_{x\to\infty} (\ln(x))^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln(\ln x)^{\frac{1}{x}}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln(\ln x)} = \lim_{x\to\infty} e^{\frac{\ln(\ln x)}{x}}$

"∞^0"

$= e^{\lim_{x\to\infty} \frac{\ln(\ln x)}{x}} = e^{\lim_{x\to\infty} \frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1}} = e^{\lim_{x\to\infty} \frac{1}{x\ln x}} = e^0 = \boxed{1}$

(9) $\lim_{x\to 0^+} (\sin(x))^{\frac{1}{x}} = (\sin(0))^{\frac{1}{0}} = $ "0^∞" $= \boxed{0}$

NOT INDETERMINATE FORM

(10) $\lim_{x\to 0^+} \frac{\sin(x)}{\ln(x)} = \lim_{x\to 0^+} \frac{\cos x}{\frac{1}{x}} = \lim_{x\to 0^+} x\cos x = \boxed{0}$

"$\frac{0}{\infty}$"

(11) $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0^+} \left(\frac{x\sin x - x}{x\sin x}\right) = \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x}$

"$\frac{0}{0}$" "$\frac{0}{0}$"

$= \lim_{x\to 0^+} \frac{-\sin x}{2\cos x - x\sin x}$

$= \boxed{0}$

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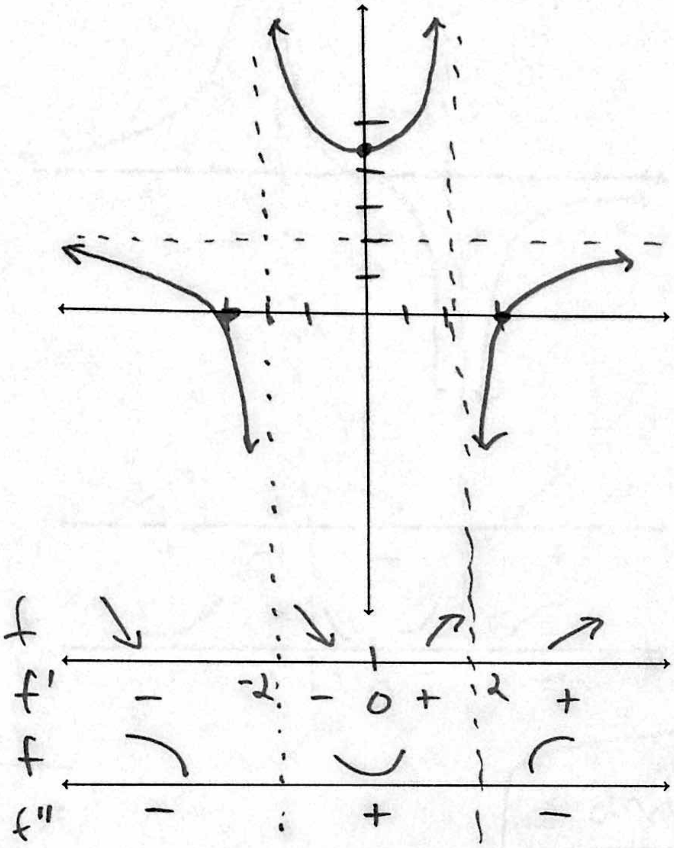
Function Analysis/Graphing

Sketch the graph of the following functions.

(a) $f(x) = \dfrac{2(x^2-9)}{x^2-4}$



1st Derivative

$$f'(x) = \frac{4x(x^2-4) - 2(x^2-9)2x}{(x^2-4)^2}$$

$$= \frac{\cancel{4x^3} - 16x - \cancel{4x^3} + 36x}{(x^2-4)^2}$$

$$= \frac{20x}{(x^2-4)^2} \rightarrow x=0$$

always + $\rightarrow (x^2-4)^2 \rightarrow x=\pm 2$

2nd Derivative

$$f''(x) = \frac{\left[20(x^2-4)^{\cancel{2}} - 20x \cdot 2\cancel{(x^2-4)} \cdot 2x\right]}{(x^2-4)^{\cancel{4}3}}$$

$$= \frac{20x^2 - 80 - 80x^2}{(x^2-4)^3}$$

$$= \frac{-60x^2-80}{(x^2-4)^3}$$

$x = \pm 2$

always −, never 0

x-ints:

$0 = 2(x^2-9)$

$x = \pm 3$

y-int:

$$y = \frac{2(0^2-9)}{0^2-4}$$

$$= \frac{9}{2}$$

vert asymptotes

$0 = x^2 - 4$

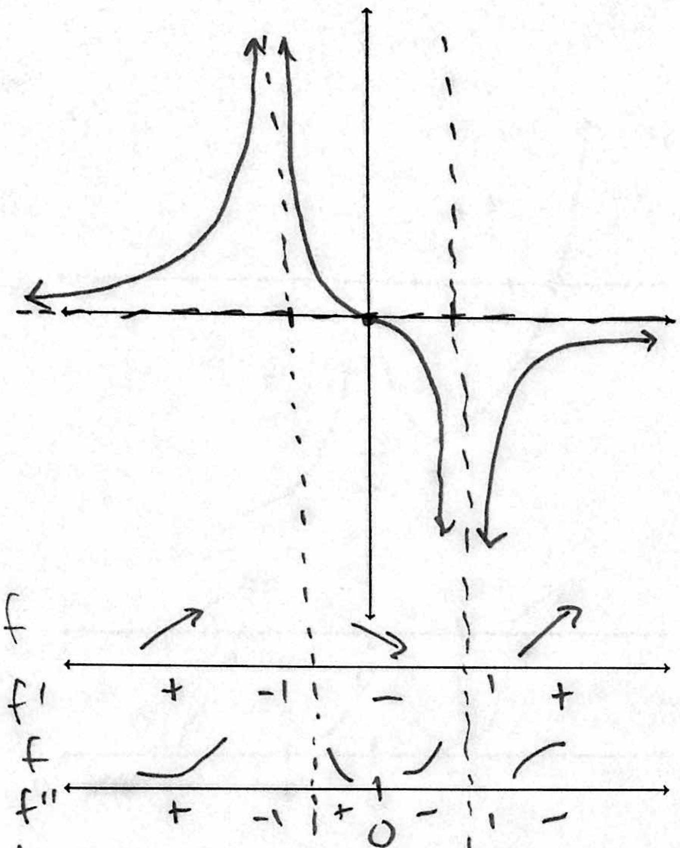
$x = \pm 2$

horizontal asymptotes

$$\lim_{x\to\infty} \frac{2(x^2-9)}{x^2-4} = 2$$

$$\lim_{x\to-\infty} \frac{2(x^2-9)}{x^2-4} = 2$$

(b) $g(x) = \frac{-x}{(x^2-1)^2}$



x-ints:

$0 = -x$

$x = 0$

y-int:

$y = 0$

vert asymptote

$0 = (x^2-1)^2$

$x = \pm 1$

horiz asymptote

$$\lim_{x\to\infty} \frac{-x}{(x^2-1)^2} = 0$$

$$\lim_{x\to-\infty} \frac{-x}{(x^2-1)^2} = 0$$

1st Derivative:

$$g'(x) = \frac{-(x^2-1)^{\cancel{2}} + x\cdot 2\cancel{(x^2-1)}\cdot 2x}{(x^2-1)^{\cancel{4}\,3}}$$

$$= \frac{-x^2+1+4x^2}{(x^2-1)^3} \rightarrow x^2-1=0$$

$x = \pm 1$

$0 = -x^2+4x^2+1$

$= 3x^2+1$

always +, never 0

2nd Derivative:

$$g''(x) = \frac{6x(x^2-1)^{\cancel{3}} - (3x^2+1)\cdot 3\cancel{(x^2-1)^2}\cdot 2x}{(x^2-1)^{\cancel{6}\,4}}$$

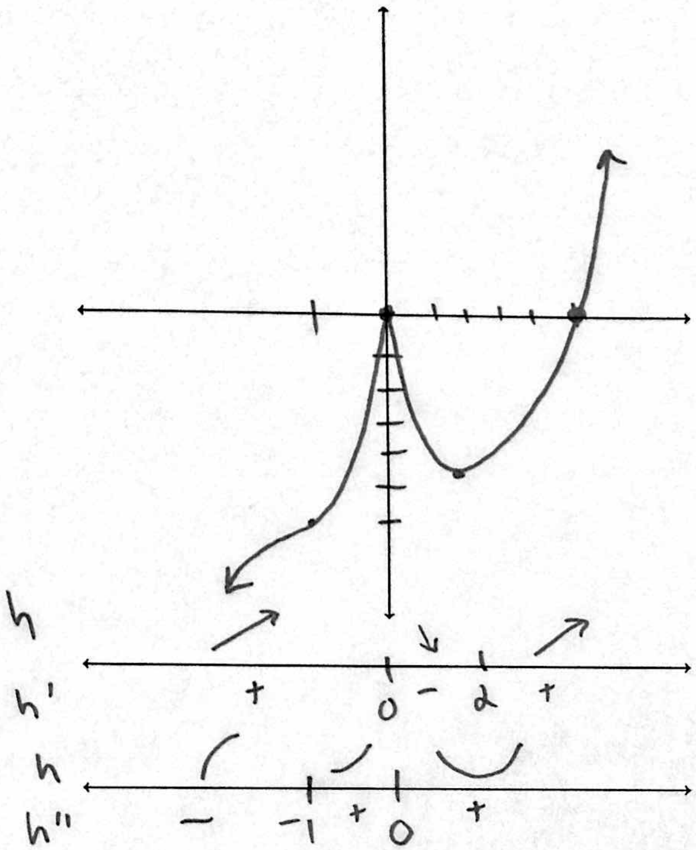
$$= \frac{6x^3-6x-18x^3-6x}{(x^2-1)^4}$$

$$= \frac{-12x^3-12x}{(x^2-1)^4} \rightarrow 0 = -12x(x^2+1)$$

$x=0$ | none

always +

(c) $h(x) = x^{5/3} - 5x^{2/3}$



x-int:

$0 = x^{5/3} - 5x^{2/3}$

$= x^{2/3}(x-5)$

$x=0 \quad | \quad x=5$

y-int:

$y=0$

vert asymp

none

horiz asymp

none

1st Derivative:

$h'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3}$

$= \frac{5x^{2/3}}{3} - \frac{10}{3x^{1/3}}$

$= \frac{5x-10}{3x^{1/3}}$ $\rightarrow x=2$, $\rightarrow x=0$

$h(2) \approx -4.76$

2nd Derivative

$h''(x) = \frac{5 \cdot 3x^{1/3} - (5x-10)\cdot x^{-2/3}}{(3x^{1/3})^2}$

$= \frac{15x^{1/3} - \frac{5x-10}{x^{2/3}}}{9x^{2/3}}$

$= \frac{15x - 5x + 10}{9x^{4/3}}$

$= \frac{10x+10}{9x^{4/3}}$ $\rightarrow x=-1$, $\rightarrow x=0$

$h(-1) = -6$

Optimization

13. Find two positive numbers such that their product is 192 and the sum of the first and three times the second is as small as possible.

$xy = 192 \Rightarrow y = \frac{192}{x}$

$T = x + 3y$

$T = x + 3\left(\frac{192}{x}\right) = x + \frac{576}{x}$

$T' = 1 - \frac{576}{x^2}$

$1 - \frac{576}{x^2} = 0$

$1 = \frac{576}{x^2}$

$x^2 = 576$

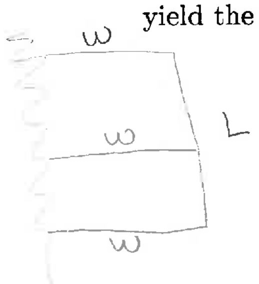
$x = \pm\sqrt{576} = \pm 24$ (has to be positive)

T ↘ ↗ min!!

T′ 0 − 24 +

$\boxed{x = 24}$ $\boxed{y = \frac{192}{24} = 8}$

14. A farmer has 500 feet of fencing with which to enclose a pasture for grazing nuggets. The farmer only need to enclose 3 sides of the pasture since the remaining side is bounded by a river (no, nuggets can't swim). In addition, some of the nuggets don't get along with some of the other nuggets. He plans to separate the troublesome nuggets by forming 2 adjacent corrals. Determine the dimensions that would yield the maximum area for the pasture.



$3w + L = 500$

$L = 500 - 3w$

$A = L \times w = (500 - 3w)w$

$= 500w - 3w^2$

$A' = 500 - 6w$

$500 - 6w = 0$

$500 = 6w$

$\frac{250}{3} = w$

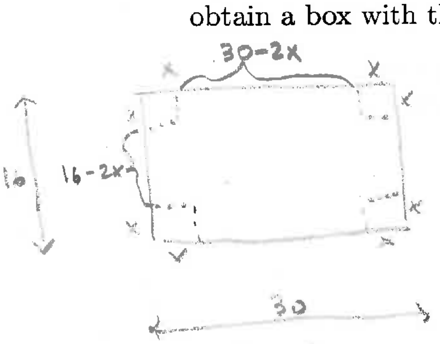
A ↗ ↘

A′ + $\frac{250}{3}$ − max!

$w = \frac{250}{3}$ ft

$L = 500 - 3\left(\frac{250}{3}\right) = 250$ ft

15. An open box is to be made from a 16-inch by 30-inch piece of cardboard by cutting out squares of equal size from each of the four corners and bending up the sides. What size should the squares be to obtain a box with the largest volume?



$0 < x < 8$ in order to form a box

$V = L \times W \times H$

$= (30 - 2x)(16 - 2x)x$

$= 480x - 92x^2 + 4x^3$

$V' = 480 - 184x + 12x^2$

$4(120 - 46x + 3x^2) = 0$

$4(3x^2 - 46x + 120) = 0$

$4(3x - 10)(x - 12) = 0$

$x = \frac{10}{3}, \cancel{12}$ not possible to form the box!

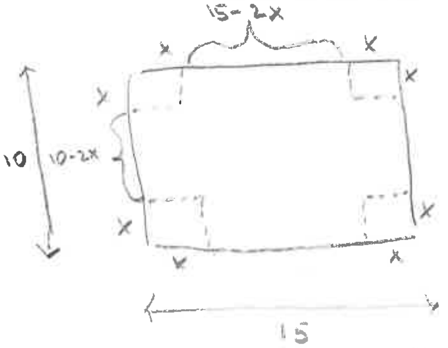
V ↗ ↘ ↗

V′ + $\frac{10}{3}$ − 12 +

max!

Squares should be $\frac{10}{3}$ in on each side to maximize volume

16. A piece of cardboard that is 10×15 (each measured in inches) is being made into a box without a top. To do so, squares are cut from each corner of the box and the remaining sides are folded up. If the box needs to be at least $\boxed{\text{1 inch deep and no more than 3 inches deep,}}$ what is the maximum possible volume of the box? what is the minimum volume? Justify your answers using calculus.



$1 \le x \le 3$

$V = L \cdot W \cdot H$

$V = (15-2x)(10-2x)x$

$V = 150x - 50x^2 + 4x^3$

$V' = 150 - 100x + 12x^2$

$2(6x^2 - 50x + 75) = 0$

$x = \frac{-(-50) \pm \sqrt{(-50)^2 - 4(6)(75)}}{2(6)} = \frac{50 \pm \sqrt{700}}{12} = \frac{50 \pm 10\sqrt{7}}{12} = \frac{25 \pm 5\sqrt{7}}{6}$

crit #'s: $x = \frac{25 \pm 5\sqrt{7}}{6}$

$x \approx 1.96, 6.37$

only $x = \frac{25 - 5\sqrt{7}}{6} \approx 1.96$ falls in $1 \le x \le 3$, so check that + endpts:

$V(1.96) \approx 132.038 \text{ in}^3$

$V(1) = (15-2)(10-2)(1) = 13 \cdot 8 \cdot 1 = 104 \text{ in}^3$

$V(3) = (15-6)(10-6)(3) = (9)(4)(3) = 108 \text{ in}^3$

$\boxed{\text{min vol} = 104 \text{ in}^3 \quad \text{max vol} \approx 132.038 \text{ in}^3}$

17. A soup can in the shape of a right circular cylinder is to be made from two materials. The material for the side of the can costs \$0.015 per square inch and the material for the lids costs \$0.027 per square inch. Suppose that we desire to construct a can that has a volume of 16 cubic inches. What dimensions minimize the cost of the can?

$V = \pi r^2 h = 16 \Rightarrow h = \frac{16}{\pi r^2}$

Surface Area $= \underbrace{2(\pi r^2)}_{\text{top + bottom}} + \underbrace{2\pi r h}_{\text{side}}$

$= 2\pi r^2 + 2\pi r\left(\frac{16}{\pi r^2}\right)$

$= 2\pi r^2 + \frac{32}{r}$

Cost $= .027(2\pi r^2) + .015\left(\frac{32}{r}\right)$

$= .054\pi r^2 + \frac{.48}{r}$

$C' = .108\pi r - \frac{.48}{r^2}$

$.108\pi r - \frac{.48}{r^2} = 0$

$.108\pi r = \frac{.48}{r^2}$

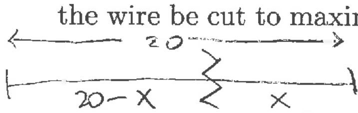
$r^3 = \frac{.48}{.108\pi}, \quad r^3 = \frac{40}{9\pi}, \quad r = \sqrt[3]{\frac{40}{9\pi}} \approx 1.1226$

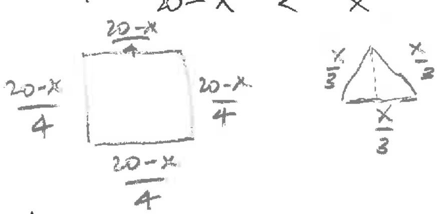
C, C': 0, $-$, 1.12, $+$

min cost occurs when $r = \sqrt[3]{\frac{40}{9\pi}} \approx 1.1226 \text{ in}$

$h = \frac{16}{\pi\left(\sqrt[3]{\frac{40}{9\pi}}\right)^2} \approx .878 \text{ in}$

18. A 20 cm piece of wire could be cut into two pieces. One piece is used to form a square and the other an equilateral triangle or all 20 cm could be used for just the square or just the triangle. How should the wire be cut to maximize the total area enclosed by the square and triangle? to minimize the area? max





$0 \le x \le 20$

$A_s = \left(\frac{20-x}{4}\right)\left(\frac{20-x}{4}\right) = \frac{(20-x)^2}{16}$

$A_T = \frac{1}{2}\left(\frac{x}{3}\right)\sqrt{\frac{x^2}{9} - \frac{x^2}{36}} = \frac{x}{3}\sqrt{\frac{4x^2 - x^2}{36}}$

$= \left(\frac{x}{6}\right)\left(\frac{\sqrt{3x^2}}{6}\right) = \frac{x^2\sqrt{3}}{36}$

Total Area $= \frac{400 - 40x + x^2}{16} + \frac{x^2\sqrt{3}}{36}$

$\frac{d(\text{Total Area})}{dx} = \frac{-40 + 2x}{16} + \frac{x\sqrt{3}}{18}$

$-\frac{5}{2} + \frac{1}{8}x + \frac{x\sqrt{3}}{18} = 0$

$x\left(\frac{9 + 4\sqrt{3}}{72}\right) = \frac{5}{2}$

$x = \frac{5}{2}\left(\frac{72}{9 + 4\sqrt{3}}\right) \approx 11.3007$

A, A': 0, $-$, 11.3, $+$, 20

$A(0) = \frac{400}{16} = 25 \text{ cm}^2$

$A(20) = \frac{20^2\sqrt{3}}{36} = \frac{100\sqrt{3}}{9} \approx 19.24 \text{ cm}^2$

$A(11.3007) \approx 10.874 \text{ cm}^2$ min

min: 11.3007 cm triangle, 8.6993 cm square

max: 0 cm triangle, 20 cm square

Implicit Differentiation

(19) Determine the slope of the tangent line to $4xy + 2y^2 - x = 2x^3 - 24$ at the point $(2, -1)$

$$\frac{d}{dx}(4xy + 2y^2 - x) = \frac{d}{dx}(2x^3 - 24)$$

$$4y + 4x\frac{dy}{dx} + 4y\frac{dy}{dx} - 1 = 6x^2$$

$$\frac{dy}{dx}(4x + 4y) = 6x^2 - 4y + 1$$

$$\frac{dy}{dx} = \frac{6x^2 - 4y + 1}{4x + 4y}$$

So the slope of the tangent line at $(2,-1)$ is:

$$\left.\frac{dy}{dx}\right|_{(2,-1)} = \frac{6(2^2) - 4(-1) + 1}{4(2) + 4(-1)}$$

$$= \boxed{\frac{29}{4}}$$

(20) Determine $\frac{dy}{dx}$ if $x^2 + y^3 = e^x \cos(y)$

$$\frac{d}{dx}(x^2 + y^3) = \frac{d}{dx}(e^x \cos y)$$

$$2x + 3y^2\frac{dy}{dx} = e^x \cos y + e^x\left(-\sin y \cdot \frac{dy}{dx}\right)$$

$$3y^2\frac{dy}{dx} + e^x \sin y \frac{dy}{dx} = e^x \cos y - 2x$$

$$\frac{dy}{dx}(3y^2 + e^x \sin y) = e^x \cos y - 2x$$

$$\boxed{\frac{dy}{dx} = \frac{e^x \cos y - 2x}{3y^2 + e^x \sin y}}$$

(21) Determine $\frac{dy}{dx}$ if $x^y = y^x$.

Assuming $x > 0, y > 0$:

$$x^y = y^x$$

$$\ln(x^y) = \ln(y^x)$$

$$y \ln x = x \ln y$$

$$\frac{d}{dx}(y \ln x) = \frac{d}{dx}(x \ln y)$$

$$\frac{dy}{dx} \cdot \ln x + y \cdot \frac{1}{x} = \ln y + x \cdot \frac{1}{y}\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\ln x - \frac{x}{y}\right) = \ln y - \frac{y}{x}$$

$$\boxed{\frac{dy}{dx} = \frac{\ln y - y/x}{\ln x - x/y}}$$

22. For the curve that satisfies the equation $2(x^2+y^2)^2 = 25(x^2-y^2)$ that is shown below

```
In[1]:= ContourPlot[2*(x^2+y^2)^2 == 25*(x^2-y^2), {x, -4, 4}, {y, -2, 2},
 ContourStyle → {Red, Thick}]
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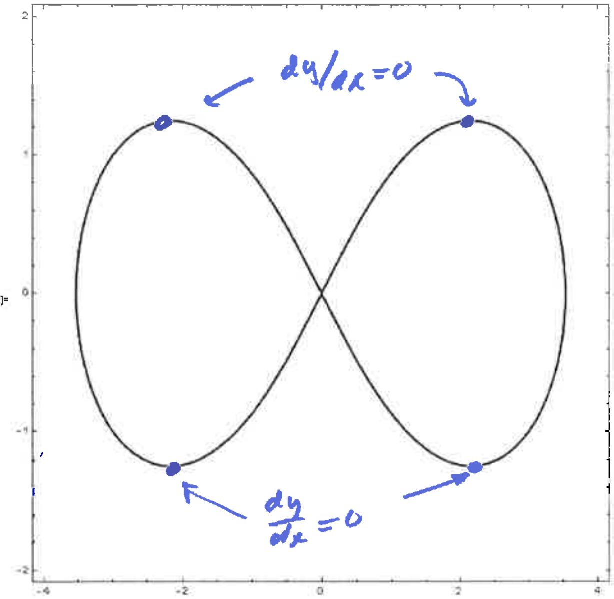
Out[1]=



(a) Determine $\frac{dy}{dx}$.

(b) Identify all point where $\frac{dy}{dx}=0$ on the graph and then determine their coordinates algebraically.

(c) The tangent lines to the curve at $x=3$ intersect on the x axes. Determine their point of intersection.

a) $\frac{d}{dx}\left(2(x^2+y^2)^2\right) = \frac{d}{dx}\left(25(x^2-y^2)\right)$

$$4(x^2+y^2)\cdot\left(2x+2y\frac{dy}{dx}\right) = 50x - 50y\frac{dy}{dx}$$

$$8x(x^2+y^2) + 8y(x^2+y^2)\frac{dy}{dx} = 50x - 50y\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(8y(x^2+y^2)+50y\right) = 50x - 8x(x^2+y^2)$$

$$\boxed{\frac{dy}{dx} = \frac{50x-8x(x^2+y^2)}{50y+8y(x^2+y^2)}}$$

b) $0 = \frac{dy}{dx} = \frac{50x-8x(x^2+y^2)}{50y+8y(x^2+y^2)}$

Note: at $(0,0)$ $\frac{dy}{dx}$ is not defined. Looking at the graph can you see why that might be?

$$0 = 50x - 8x(x^2+y^2)$$

$x=0$ OR $0 = 50-8(x^2+y^2)$

Not a soln by the note.

$$\frac{50}{8} = x^2+y^2$$

$$\boxed{x^2 = \frac{50}{8} - y^2}$$

So the points on the curve where $\frac{dy}{dx}=0$ also satisfy $x^2 = y^2 + \frac{50}{8}$. To find the points substitute into the original equation.

The original equation $2(x^2+y^2)^2 = 25(x^2-y^2)$ becomes:

$$2\left(\frac{50}{8}-y^2+y^2\right)^2 = 25\left(\frac{50}{8}-y^2-y^2\right)$$

$$\frac{625}{8} = \frac{1250}{8} - 50y^2$$

$$\frac{625}{8} = 50y^2$$

$$\frac{625}{400} = y^2$$

$$\pm\frac{5}{4} = y$$

Now to find x: $x^2 = \frac{50}{8} - y^2 = \frac{50}{8} - \frac{25}{16} = \frac{75}{16}$

$$x = \pm\frac{5\sqrt{3}}{4}$$

So the points where $\frac{dy}{dx} = 0$ are: $\left(\frac{5\sqrt{3}}{4}, \frac{5}{4}\right), \left(\frac{5\sqrt{3}}{4}, -\frac{5}{4}\right), \left(-\frac{5\sqrt{3}}{4}, \frac{5}{4}\right), \left(-\frac{5\sqrt{3}}{4}, -\frac{5}{4}\right)$

c) We need to find the points on the curve when $x = 3$ and then evaluate $\frac{dy}{dx}$ at those points to get the slope of the tangent lines.

$$2(3^2+y^2)^2 = 25(3^2-y^2)$$

$$2(81+18y^2+y^4) = 25(9-y^2)$$

$$2y^4+61y^2-63 = 0$$

$$(2y^2+63)(y^2-1) = 0$$

$2y^2+63=0$ or $y^2-1=0$

No solutions $\qquad y = \pm 1$

So the two points are $(3,1), (3,-1)$

The slopes are:

$$\left.\frac{dy}{dx}\right|_{(3,1)} = \frac{50(3)-8(3)(3^2+1^2)}{50(1)+8(1)(3^2+1^2)} = \frac{-90}{130} = \frac{-9}{13}$$

Similarly $\left.\frac{dy}{dx}\right|_{(3,-1)} = \frac{9}{13}$

Since the slope of tangent line is $-\frac{9}{13}$ and we start from the point $(3,1)$ then to move down by 1 we must move right by $13/9$ to hit the point $\left(3+\frac{13}{9}, 0\right) = \left(\frac{40}{9}, 0\right)$

Similarly the tangent line at $(3,-1)$ goes through $\left(\frac{40}{9}, 0\right)$ and hence this is their point of intersection.

Related Rates

23 ☐ Suppose x and y are differentiable functions of t and are related by $y = x^2 - 1$. Find dy/dt when $x = 2$ given that $dx/dt = 3$.

$$y = x^2 - 1$$

$$\frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dy}{dt} = 2(2)(3) = \boxed{12}$$

24 ☐ A nugget is dropped into a calm pond, causing concentric circles. The radius of the outer ripple is increasing at a rate of 2 ft/sec. When the radius is 3 feet, at what rate is the total area of the outer ripple changing?

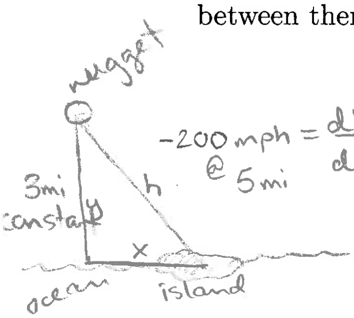
$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

$$= 2\pi(3)(2) = 12\pi \ \frac{ft^2}{sec}$$

The area of the outer ripple increasing at a rate of $12\pi \ \frac{ft^2}{sec}$

25 ☐ A nugget is flying on a flight path 3 miles above the ocean that will take it directly over an island. If the distance between the nugget and island is decreasing at a rate of 200 mph when the distance between them is 5 miles, what is the speed of the nugget?



$-200 \text{ mph} = \frac{dh}{dt}$ @ 5mi

$\frac{dx}{dt} = ?$

3-4-5 triangle so $x = 4$mi

$$x^2 + y^2 = h^2$$

constant

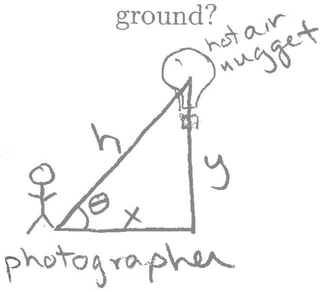
$$2x\frac{dx}{dt} = 2h\frac{dh}{dt}$$

$$4\left(\frac{dx}{dt}\right) = 5(-200)$$

$$\frac{dx}{dt} = -250 \text{ mph}$$

the nuggets speed is 250 mph

26. A hot-air nugget is rising at a rate of 15 ft/sec when the nugget is 50 ft off the ground. A photographer is standing on the ground 100 feet from the take-off site. If the photographer keeps the nugget in sight, what is the rate of change in the photographer's angle of elevation when the nugget is 50 feet off the ground?



$\frac{dy}{dt} = 15 \frac{ft}{sec}$

when $y = 50$ ft

$x = 100$ ft constant!

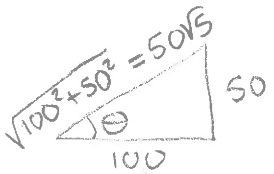
$\frac{d\theta}{dt} = ?$

$\tan\theta = \frac{y}{x}$

$\tan\theta = \frac{y}{100}$

$100\tan\theta = y$

$100\sec^2\theta \frac{d\theta}{dt} = \frac{dy}{dt}$



$\sec\theta = \frac{50\sqrt{5}}{100} = \frac{\sqrt{5}}{2}$

$100 \cdot \left(\frac{\sqrt{5}}{2}\right)^2 \frac{d\theta}{dt} = 15$

$\frac{d\theta}{dt} = \frac{3}{25} \frac{ft}{sec}$

$\boxed{\text{The photographer's angle of elevation is increasing by } \frac{3}{25}\frac{ft}{sec}}$

27. A spherical nugget is being inflated with air, so that the volume is increasing at a rate of 3 cubic meters per minute. Find the rate of change of the radius when the radius is 5 meters.

$V = \frac{4}{3}\pi r^3$ $\quad \frac{dV}{dt} = 3 \frac{m^3}{min}$ $\quad \frac{dr}{dt} = ?$ $\quad r = 5$ m

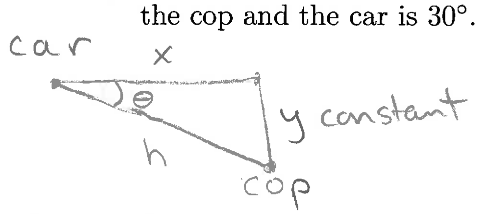
$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$3 = 4\pi(5)^2 \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{3}{100\pi} \frac{m}{min}$

The rate of change of the radius in $\frac{3}{100\pi} \frac{m}{min}$

28. A cop with a radar gun is by the side of the road. The gun measures that a car is approaching the police officer at 50 mph. At that moment the angle between the road and the line of sight between the cop and the car is 30°. Is the car violating the 55 mph speed limit?



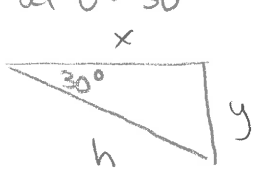
$x^2 + y^2 = h^2$

$2x\frac{dx}{dt} + 0 = 2h\frac{dh}{dt}$

$\frac{dh}{dt} = -50$ mph $\quad \frac{dx}{dt} = ?$

$\frac{dx}{dt} = \frac{h}{x}\frac{dh}{dt}$

at $\theta = 30°$



$\cos 30° = \frac{x}{h}$

so $\frac{h}{x} = \frac{1}{\cos 30°} = \frac{2}{\sqrt{3}}$

$\frac{dx}{dt} = \frac{2}{\sqrt{3}} \cdot (-50) = -57.7$ mph

$\boxed{\text{The car is violating the speed limit since they are going 57.7 mph}}$

Mean Value Theorem

29. Suppose a car that is equipped with an E-Z Pass drives from the toll plaza in Carlisle, PA (milage marker 226 on the PA Turnpike) to the one in Valley Forge (marker 326) in 1 hour and 15 minutes. A few days later the driver receives a speeding ticket in the mail. How did the PA state troopers know that the driver was speeding?

$a = 0$ hrs $f(a) = 226$ miles

$b = 1.25$ hrs $f(b) = 326$ miles

f is distance in miles

f' is velocity in miles/hr

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{326-226}{1.25-0} = 80 \text{ mi/hr}$$

At some point she was traveling 80mi/hr so she was speeding.

30. Consider the function $f(x) = \frac{x}{x+2}$.

(a) Verify that f satisfies the hypotheses of the Mean Value Theorem on the interval $[1, 4]$ and then find all of the values, c, that satisfy the conclusion of the theorem.

f is not continuous at $x=-2$, $x=-2$ is not in the interval so it is OK

f is not differentiable at $x=-2$ but $x=-2$ is not in the inverval so it is OK

$$f(1) = \frac{1}{1+2} = \frac{1}{3}$$

$$f(4) = \frac{4}{4+2} = \frac{2}{3}$$

$$f'(x) = \frac{1(x+2)-x(1)}{(x+2)^2} = \frac{2}{(x+2)^2}$$

$$\frac{2}{(c+2)^2} = \frac{\frac{2}{3}-\frac{1}{3}}{4-1}$$

$$\frac{2}{(c+2)^2} = \frac{1}{9}$$

$$18 = (c+2)^2$$

$c = -2 \pm \sqrt{18}$

$-2-\sqrt{18} \notin [-2, 6]$

$\boxed{c = -2+\sqrt{18}}$

(b) Why does f not satisfy the hypothesis of the Mean Value Theorem on the interval $[-8, 6]$

f is not continuous or differentiable at $x = -2$ which is in the interval $[-8, 6]$