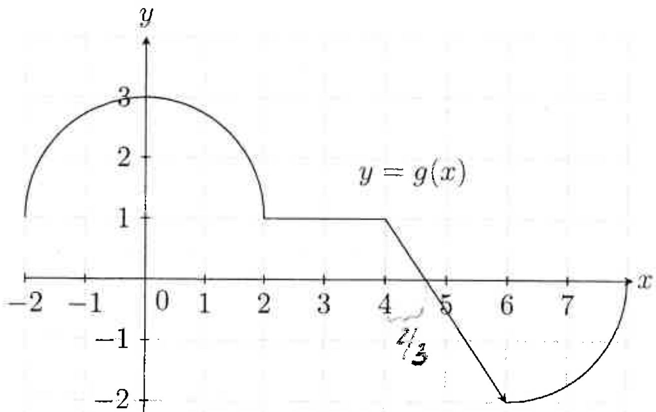
Intuitive Definite Integral

1.



(a) Find $\int_0^4 g(x)\,dx$.

$\frac{1}{4}\pi(2)^2 + 1(4) = \boxed{4+\pi}$

(b) Find $\int_{-2}^8 g(x)\,dx$.

$\frac{1}{2}\pi(2)^2 + 1(6) + \frac{1}{2}\left(\frac{2}{3}\right)(1) - \frac{1}{2}\left(\frac{4}{3}\right)(2) - \frac{1}{4}\pi(2)^2$

$\pi + 6 + \frac{1}{3} - \frac{4}{3}$

$\boxed{\pi + \frac{16}{3}}$

$y-1=\frac{-3}{2}(x-4)$

$0-1=\frac{-3}{2}x+6$

$-7=\frac{-3}{2}x$

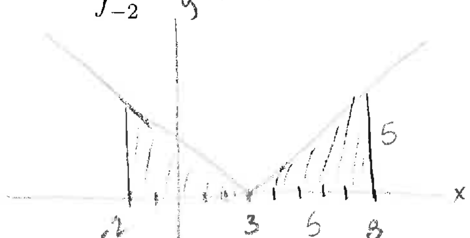
$x=\frac{14}{3}$

$\frac{14}{3}-\frac{12}{3}=\frac{2}{3}$

2. Use area of basic geometric shapes to find the following definite integrals.

(a) Find $\int_{-2}^8 |x-3|\,dx$.

$2\left(\frac{1}{2}\right)(5)(5) = \boxed{25}$



(b) Find $\int_{-3}^3 -\sqrt{9-x^2}\,dx$.

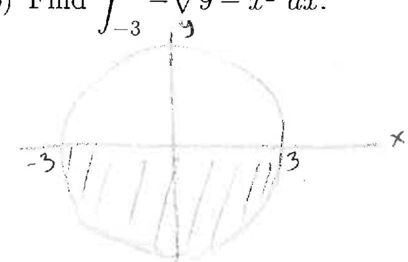
$y=-\sqrt{9-x^2}$

$y^2=9-x^2$

$x^2+y^2=9$

$-\frac{1}{2}\pi(3)^2$

$=\boxed{-\frac{9}{2}\pi}$



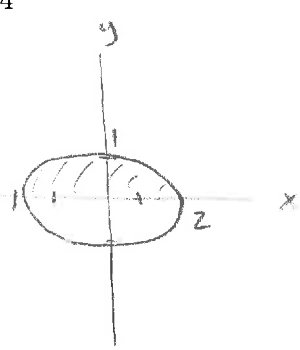
(c) Find $\int_0^2 \sqrt{1-\frac{x^2}{4}}\,dx$.

$y=\sqrt{1-\frac{x^2}{4}}$

$y^2=1-\frac{x^2}{4}$

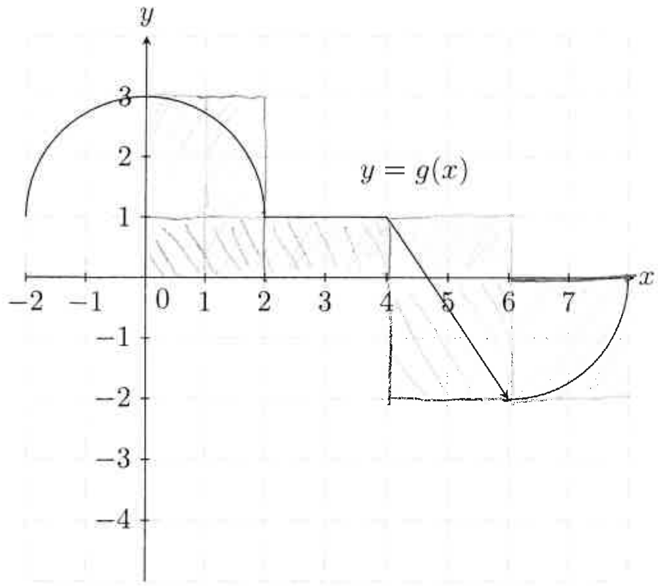
$\frac{x^2}{4}+y^2=1$

$\frac{1}{2}\pi(2)(1) = \boxed{\pi}$



Riemann Sums

3. Estimate $\int_0^8 g(x)\,dx$ using 4 intervals and:



(a) left end points.

$2[3+1+1-2]$

$= \boxed{6}$

(b) right end points.

$2[1+1-2+0] = \boxed{0}$

4. Find the left and the right sum of $f(x)=\sqrt{x}+2$ on the interval [0,1] using 5 subdivisions.

$L_0^1(f,5) = \frac{1}{5}\left[(\sqrt{0}+2)+(\sqrt{1/5}+2)+(\sqrt{2/5}+2)+(\sqrt{3/5}+2)+(\sqrt{4/5}+2)\right] \approx 2.55$

$R_0^1(f,5) = \frac{1}{5}\left[(\sqrt{1/5}+2)+(\sqrt{2/5}+2)+(\sqrt{3/5}+2)+(\sqrt{4/5}+2)+\sqrt{1}+2\right] \approx 2.75$

5. The following sum: $3(\sqrt{5}+1)+3(\sqrt{8}+1)+3(\sqrt{11}+1)+3(\sqrt{14}+1)$ is a right Riemann sum for a certain definite integral $\int_2^b f(x)\,dx$ using a partition of the interval $[2,b]$ into 4 subintervals of equal length.

2 5 8 11 b=14

(a) What is b? $\boxed{14}$

(b) What is $f(x)$? $\boxed{\sqrt{x}+1}$

6. The following sum: $\frac{1}{1+\frac{2}{n}}\cdot\frac{2}{n}+\frac{1}{1+\frac{4}{n}}\cdot\frac{2}{n}+\frac{1}{1+\frac{6}{n}}\cdot\frac{2}{n}+\cdots+\frac{1}{1+\frac{2n}{n}}\cdot\frac{2}{n}$ is a right Riemann sum for a certain definite integral $\int_1^b f(x)\,dx$ using a partition of the interval $[1,b]$ into n subintervals of equal length.

$\frac{2}{n} = \frac{b-a}{n} = \frac{b-1}{n} \Rightarrow b=3$

(a) What is b? $\boxed{3}$

(b) What is $f(x)$?

if $b=3$ $\quad \frac{1}{1+\frac{2n}{n}} = \frac{1}{3} = f(b)$ $\quad \boxed{f(x)=\frac{1}{x}}$

Fundamental Theorem of Calculus

7. Explain why the Fundamental Theorem of Calculus cannot be used to evaluate $\int_{-1}^{1} \frac{1}{x^2}\, dx$.

The FTC requires the integrand to be continuous on the interval in question. However, $f(x) = \frac{1}{x^2}$ is not continuous at $x=0$, which is in the interval $[-1,1]$.

8. Compute each of the following definite integrals.

(a) Let $A(x) = \int_0^x t^2 - t\, dt$. Find A'.

$A'(x) = x^2 - x$

(b) Let $f(x) = \int_0^x \sqrt[3]{t^2+1}\, dt$. Find f'.

$f'(x) = \sqrt[3]{x^2+1}$

(c) Let $G(x) = \int_0^{x^2} t^3 \sin(t)\, dt$. Find G'.

$G'(x) = (x^2)^3 \sin(x^2) \cdot 2x$ (by FTC and chain rule)

(d) Let $C(x) = \int_x^{x^3} \cos(\cos(t))\, dt$. Find C'.

$C'(x) = \cos(\cos(x^3)) \cdot 3x^2 - \cos(\cos(x))$

9. Let $A(x) = \int_0^x \sin^2 t\, dt$. Determine where A attains a maximum value on the interval $[0, \pi]$.

$A'(x) = \sin^2(x)$

$0 = \sin^2(x)$

critical numbers in $[0,\pi]$:

$x = 0, \pi$

$A(0) = \int_0^0 \sin^2(t)\,dt = 0$

$A(\pi) = \int_0^\pi \sin^2(t)\,dt > 0$

since $\sin^2(t) > 0$ on $(0,\pi)$.

$\therefore$ A attains max at $x = \pi$.

10 Definite Integrals

(a) $\int_0^1 x^2\, dx$

$= \frac{x^3}{3}\Big|_0^1$

$= \frac{1}{3} - \frac{0}{3} = \boxed{\frac{1}{3}}$

(b) $\int_{-1}^1 x^4 - \frac{1}{2}x^3 + \frac{1}{4}x - 2\, dx$

$= \frac{x^5}{5} - \frac{x^4}{8} + \frac{x^2}{8} - 2x\Big|_{-1}^1$

$= \frac{1}{5} - \cancel{\frac{1}{8}} + \cancel{\frac{1}{8}} - 2 - \left(\frac{-1}{5} - \cancel{\frac{1}{8}} + \cancel{\frac{1}{8}} + 2\right) = \boxed{\frac{-18}{5}}$

(c) $\int_0^\pi \sin(x)\, dx$

$= -\cos(x)\Big|_0^\pi$

$= -\cos(\pi) - (-\cos(0))$

$= -(-1) + 1 = \boxed{2}$

(d) $\int_0^\pi \cos(2x)\, dx$

$u = 2x$

$du = 2dx$

$dx = \frac{du}{2}$

$= \int_{x=0}^{x=\pi} \frac{\cos(u)}{2}\, du$

$= \frac{1}{2}\sin(2x)\Big|_0^\pi = \frac{1}{2}\left[\sin(2\pi) - \sin(0)\right] = \boxed{0}$

(e) $\int_0^{\ln 2} e^{x/3}\,dx = \int_{x=0}^{x=\ln 2} 3e^u\,du = 3e^u \Big|_{x=0}^{x=\ln 2} = 3e^{x/3}\Big|_0^{\ln 2} = 3e^{\ln 2/3} - 3e^{0/3}$

$= 3\cdot 2^{1/3} - 3$

Let $u = \frac{x}{3}$

$3\,du = dx$

(f) $\int_1^{e^2} \frac{x+1}{x^2}\,dx$

$= \int_1^{e^2} \frac{x}{x^2} + \frac{1}{x^2}\,dx$

$= \int_1^{e^2} \frac{1}{x} + x^{-2}\,dx$

$= \ln|x| - \frac{1}{x}\Big|_1^{e^2}$

$= \ln|e^2| - \frac{1}{e^2} - (\ln|1| - 1)$

$= 2 - \frac{1}{e^2} + 1 = \boxed{3 - \frac{1}{e^2}}$

(g) $\int_1^2 \frac{x^3 - 2\sqrt{x}}{x}\,dx$

$= \int_1^2 \frac{x^3}{x} - \frac{2x^{1/2}}{x}\,dx$

$= \int_1^2 x^2 - 2x^{-1/2}\,dx$

$= \frac{x^3}{3} - 2\cdot 2\,x^{1/2}\Big|_1^2$

$= \frac{8}{3} - 4\sqrt{2} - \left(\frac{1}{3} - 4\right)$

$= \boxed{\frac{19}{3} - 4\sqrt{2}}$

(h) $\int_0^{1/2} \frac{4}{\sqrt{1-x^2}}\,dx$

$= 4\arcsin(x)\Big|_0^{1/2}$

$= 4\left[\arcsin\left(\tfrac{1}{2}\right) - \arcsin(0)\right]$

$= 4\left[\frac{\pi}{6} - 0\right] = \boxed{\frac{2\pi}{3}}$

11. Indefinite Integrals

Compute each of the following indefinite integrals.

(a) $\int 5\ dx$ $= 5x + C$

(a) $\int 0\ dx$ $= C$

(b) $\int 2x^3 + x^2 - 5x + 5\ dx$ $= \frac{1}{2}x^4 + \frac{1}{3}x^3 - \frac{5}{2}x^2 + 5x + C$

(c) $\int -2\sqrt{x}\ dx$ $= \int -2x^{\frac{1}{2}}dx = -\frac{4}{3}x^{\frac{3}{2}} + C$

(d) $\int \frac{x+1}{\sqrt{x}}\,dx = \int \left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) dx = \int \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right) dx$

$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

(e) $\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = -\frac{1}{2}x^{-2} + C$

(f) $\int \frac{x+5}{x^2}\,dx = \int \left(\frac{x}{x^2} + \frac{5}{x^2}\right) dx = \int \left(\frac{1}{x} + 5x^{-2}\right) dx = \ln|x| - 5x^{-1} + C$

(g) $\int \frac{\sin(x)}{\cos^2(x)}\,dx = \int \sec x \tan x\,dx = \sec x + C$

12. Substitution

Compute each of the following integrals.

(a) $\int (3x-1)^2\,dx$ (Do 2 ways.)

1st way (simplifying)

$= \int 9x^2 - 6x + 1\,dx$

$= \frac{9x^3}{3} - \frac{6x^2}{2} + x + C$

$= \boxed{3x^3 - 3x^2 + x + C}$

2nd way (substitution)

let $u = 3x-1$

$du = 3dx$

$dx = \frac{du}{3}$

$\int u^2 \left(\frac{du}{3}\right)$

$= \frac{1}{3}\int u^2\,du$

$= \frac{1}{3}\left(\frac{u^3}{3} + C\right)$

$\boxed{\frac{(3x-1)^3}{9} + C}$

* The two are equivalent expressions.

(b) $\int (3x-1)^{99}\,dx$

let $u = 3x-1$

$du = 3dx$

$dx = \frac{du}{3}$

$\int u^{99}\left(\frac{du}{3}\right)$

$= \frac{1}{3}\int u^{99}\,du$

$= \frac{1}{3}\left(\frac{u^{100}}{100} + C\right)$

$\boxed{\frac{(3x-1)^{100}}{300} + C}$

(c) $\int 5x^2\sqrt{x^3-2}\,dx$

let $u = x^3 - 2$

$du = 3x^2\,dx$

$dx = \frac{1}{3}\frac{du}{x^2}$

$\int 5x^2 u^{\frac{1}{2}}\left(\frac{1}{3}\frac{du}{x^2}\right)$

$= \int \frac{5}{3} u^{\frac{1}{2}}\,du$

$= \frac{5}{3}\int u^{1/2}\,du$

$= \frac{5}{3}\left(\frac{2}{3}u^{3/2} + C\right)$

$= \frac{10u^{3/2}}{9} + C$

$= \boxed{\frac{10(x^3-2)^{3/2}}{9} + C}$

(d) $\int_0^2 xe^{x^2}\,dx$

let $u = x^2$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$\int_0^2 xe^u\left(\frac{du}{2x}\right)$

$= \frac{1}{2}\int_0^2 e^u\,du$

$= \frac{1}{2}e^u\Big|_0^2$

$= \frac{1}{2}e^{2x}\Big|_0^2$

$= \frac{1}{2}e^{2(2)} - \frac{1}{2}e^{2(0)}$

$= \boxed{\frac{1}{2}e^4 - \frac{1}{2}}$

(e) $\int \sin^2(x)\cos(x)\,dx$

let $u = \sin(x)$

$du = \cos(x)\,dx$

$dx = \frac{du}{\cos(x)}$

$\int u^2 \cos(x) \frac{du}{\cos(x)}$

$= \int u^2\,du$

$= \frac{u^3}{3} + C$

$= \boxed{\frac{\sin^3(x)}{3} + C}$

(f) $\int_0^1 \frac{x}{x^2+1}\,dx$

let $u = x^2+1$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$\int_0^1 \frac{x}{u}\left(\frac{du}{2x}\right)$

$= \frac{1}{2}\int_0^1 \frac{1}{u}\,du$

$= \frac{1}{2}\ln|u|\Big|_0^1$

$= \frac{1}{2}\ln(x^2+1)\Big|_0^1$

$= \frac{1}{2}\ln(1^2+1) - \frac{1}{2}\ln(0^2+1)$

$= \frac{\ln 2}{2} - 0$

$= \boxed{\frac{\ln 2}{2}}$

(g) $\int x^2\sec^2(x^3)\,dx$

let $u = x^3$

$du = 3x^2\,dx$

$dx = \frac{du}{3x^2}$

$\int x^2 \sec^2 u \left(\frac{du}{3x^2}\right)$

$= \frac{1}{3}\int \sec^2 u\,du$

$= \frac{1}{3}(\tan u + c)$

$= \boxed{\frac{1}{3}\tan(x^3) + C}$

(h) $\int \frac{x}{x^4+1}\,dx$

let $u = x^2$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$\int \frac{x}{u^2+1}\left(\frac{du}{2x}\right)$

$= \frac{1}{2}\int \frac{1}{u^2+1}\,du$

$= \frac{1}{2}(\arctan(u) + C)$

$= \boxed{\frac{1}{2}\arctan(x^2) + C}$

(i) $\int x\sqrt{x-1}\,dx$

let $u = x-1$

$du = 1\,dx$

$\int x\, u^{\frac{1}{2}}(1\,dx)$

$= \int x\, u^{\frac{1}{2}}\,dx$

* since $u = x-1$,

$x = u+1$

$\int (u+1)u^{\frac{1}{2}}\,dx$

$= \int (u^{3/2} + u^{1/2})\,dx$

$= \frac{2u^{5/2}}{5} + \frac{2u^{3/2}}{3} + C$

$= \boxed{\frac{2(x-1)^{5/2}}{5} + \frac{2(x-1)^{3/2}}{3} + C}$

13. **Parts**

Integrate each of the following.

(a) $\int xe^{-x}\,dx = x(-e^{-x}) - \int -e^{-x}dx = \boxed{-xe^{-x} - e^{-x} + C}$

"dv" : If $f'(x)=e^{-x}$ then $f(x)=-e^{-x}$ ← "v"

"u" : If $g(x)=x$ then $g'(x)=dx$ ← "du"

For sake of brevity:

We will now use the convention "$u=g(x)$" and "$dv=f'(x)dx$" so that
$\int u\,dv = uv - \int v\,du$ (that is $\int g(x)f'(x)dx = g(x)f(x) - \int f(x)g'(x)dx$)

(b) $\int x^2\sin(x)\,dx \underset{\text{by (1)}}{=} x^2(-\cos x) - \int -\cos x(2x)dx \underset{\text{by (2)}}{=} -x^2\cos x + 2x\sin x - \int 2\sin x\,dx$

polynomial "periodically differentiable"

$= \boxed{-x^2\cos x + 2x\sin x + 2\cos x + C}$

(1) Let $u_1 = x^2$ $dv_1 = \sin x\,dx$
$du_1 = 2x\,dx$ $v_1 = -\cos x$

(2) Let $u_2 = 2x$ $dv_1 = \cos x\,dx$
$du_2 = 2\,dx$ $v_1 = \sin x$

(c) $\int \ln x\,dx = (\ln x)x - \int x\cdot\frac{1}{x}dx = x\ln x - \int 1dx = \boxed{x\ln x - x + C}$

Note $\ln x = 1\cdot\ln x$ and we know the derivative of $\ln x$ ...

So let $u=\ln x$ and $dv = 1dx$
$du = \frac{1}{x}dx$ $v = x$

(d) $\int_0^1 \arctan(x)\,dx = x\arctan x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}dx = x\arctan x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}dx$

Similar to (c): Let $u=\arctan x$ $dv = 1dx$
$du = \frac{1}{1+x^2}dx$ $v = x$

$= x\arctan x\Big|_0^1 - \frac{1}{2}\ln|1+x^2|\Big|_0^1$

$= 1\arctan 1 - 0\arctan 0 - \frac{1}{2}\left(\ln|1+1^2| - \ln|1+0^2|\right)$

$= \boxed{\frac{\pi}{4} - \frac{1}{2}\ln(2)}$

(e) $\int x^3 e^{3x}\, dx = \frac{1}{3}x^3e^{3x} - \int 3x^2 \cdot \frac{1}{3}e^{3x}dx = \frac{1}{3}x^3e^{3x} - \left(3x^2 \cdot \frac{1}{9}e^{3x} - \int 6x \cdot \frac{1}{9}e^{3x}dx\right)$

$= \frac{1}{3}x^3e^{3x} - \left(\frac{1}{3}x^2e^{3x} - \left(6x \cdot \frac{1}{27}e^{3x} - \int \frac{6}{27}e^{3x}dx\right)\right)$

$= \boxed{\frac{1}{3}x^3e^{3x} - \frac{1}{3}x^2e^{3x} + \frac{2}{9}xe^{3x} - \frac{2}{27}e^{3x} + C}$

This requires 3 integration by parts steps or equivalently a shortcut to by parts w/ the "tabular method." The steps shown are the result of the 3 by parts steps. Each time let u = polynomial and dv = exponential dx parts.

(f) $\int x^5 \sin(x^3)\, dx = \int \frac{1}{3} w \sin(w)\,dw = \frac{1}{3}\left(-w\cos w - \int -\cos w\, dw\right) = \frac{1}{3}(-w\cos w + \sin w) + C$

$= \boxed{\frac{1}{3}\left(-x^3\cos(x^3) + \sin(x^3)\right) + C}$

Let $w = x^3$

$dw = 3x^2dx$

Thus $x^5dx = \frac{1}{3}w\,dw$

Now let $u = w$ and $dv = \sin w\, dw$

$du = dw$ $\quad v = -\cos w$

Let $u_1 = e^x$ $\quad dv_1 = \cos x\, dx$

$du_1 = e^x dx$ $\quad v_1 = \sin x$

Let $u_2 = e^x$ $\quad dv_2 = \sin x\, dx$

$du_2 = e^x dx$ $\quad v_2 = -\cos x$

(g) $\int e^x \cos(x)\, dx. = e^x\sin x - \int \sin x\, e^x dx = e^x \sin x - \left(-e^x\cos x - \int -\cos x\, e^x dx\right) + C$

$= e^x \sin x + e^x\cos x - \int \cos x\, e^x dx + C$

Therefore $2\int \cos x\, e^x dx = e^x \sin x + e^x \cos x + C$

and so... $\boxed{\int \cos x\, e^x dx = \frac{1}{2}\left(e^x\sin x + e^x\cos x\right) + C}$

Falling Objects

14. A skydiver steps out of an airplane. Her velocity in feet per second in the first 15 seconds of the fall can be represented by the function $f(x) = 30(1 - e^{-x/3})$. Find the distance fallen by the skydiver after 15 seconds have passed.

Distance function $\rightarrow \int 30(1-e^{-x/3}) = 30(x+3e^{-x/3}) + C$

Initial Distance fallen after $t=0$ seconds is 0

So $D(0) = 30(0+3e^{0}) + C = 0 \rightarrow C = -90$

Thus $\boxed{D(t) = 30(x+3e^{-x/3}) - 90}$

So after 15 seconds..

$D(15) = 30(15+3e^{-15/3}) - 90$

$= 360.6$ ft is the distance fallen after 15 sec.

15. During the 2014 Flagstaff earthquake, a pinecone fell from a tree on the edge of a cliff, falling 215 meters.

(a) How long did it take the piece of pinecone to hit the ground?

Note the distance fallen after t seconds can be found by taking $\int |v(t)| = \int 9.8t = \frac{9.8t^2}{2} + C = 4.9t^2 + C$

The pinecone fell 0 meters when $t=0$ so

$D(0) = 4.9(0)^2 + C = 0 \rightarrow \boxed{D(t) = 4.9t^2} \leftarrow$ Distance function fallen

The pinecone fell 215 meters when $t=?$

$\frac{215}{4.9} = \frac{\cancel{4.9}t^2}{\cancel{4.9}} \rightarrow \boxed{t = 6.62 \text{ seconds}}$

(b) Ignoring air resistance, what will the velocity of the pinecone when it strikes the ground?

$V(t) = -9.8t$ so when $t=6.62$ $V(6.62) = -9.8(6.62)$

$= \boxed{-64.876 \frac{m}{s}}$

16. An person falls from the tallest building in Flagstaff and takes 3 seconds to reach the ground.

(a) What is its speed at impact if air resistance is ignored?

$S(t) = |v(t)| = 9.8t$

Impact when $t = 3$

So $S(3) = 9.8(3)$

$= \boxed{29.4 \frac{m}{s}}$

(b) How tall is the building?

$D(t) = 4.9t^2$

$\longrightarrow D(3) = 4.9(3^2) = \boxed{44.1 \text{ meters}}$

(c) What is the person's acceleration at the 2nd second?

Free fall acceleration is constant at $\boxed{-9.8 \text{ m/sec}^2}$