Supplemental Problems for Exam 3

General Information

Exam 3 covers material that we have covered since the first day of class up until the end of class on Wednesday, November 2. However, the exam focuses on content covered since Exam 2. In particular, here is a list of topics/concepts (since Exam 2) that you must have an understanding of or be able to do:

- Have an intuitive understanding of derivatives, based on your knowledge of rate of change, speed, velocity, and slope of tangent lines.
- Be able to solve related rates problems.
- Find the local linearization (i.e., tangent line approximation) and the equation of the tangent line to the graph of a function at a particular point.
- Evaluate limits of various indeterminate forms.
- Be able to apply L'Hôpital's Rule and know when it applies.
- Be able to state the Mean Value Theorem and be able to apply it in appropriate circumstances.
- Know the definition of critical numbers and be able to find them. *Note:* We use the terms "critical points" and "critical numbers" interchangeably.
- Understand the concept of local maximum and local minimum.
- Understand the relationship between critical points and local extrema.
- Understand the concept of global maximum and global minimum.
- Understand the relationship between the sign of the derivative and the direction of the original function (increasing versus decreasing).
- Understand the relationship between the sign of the second derivative and concavity of the original function (concave up versus concave down).
- Know the First Derivative Test and be able to use it to make conclusions about local extrema.
- Know the definition of a point of inflection and be able to find them.
- Be able to find vertical and horizontal asymptotes. For horizontal asymptotes, be prepared to use L'Hôpital's Rule.
- Be able to sketch the graph of a function following the function analysis process.
- Be able to find absolute extrema of a continuous function on a closed interval.
- Be able to give examples and counter examples to demonstrate different properties of functions.
- Call upon your own mental faculties to respond in flexible, thoughtful and creative ways to problems that may seem unfamiliar on first glance.

The problems that follow will provide you with an opportunity to review the relevant topics. However:

This is not a practice test!

It is possible that problems on your exam will resemble problems on this review, but you should not expect exam problems to be identical to ones found below. This review contains an abundance of problems and it is not the intention that every student will complete every problem. You should complete as many problems in each section below as you think are necessary to solidify your understanding.

In addition, you should review examples done in class, as well as your homework exercises, especially the ones on the Weekly Homework assignments.

Words of Advice

Here are few things to keep in mind when taking your exam:

- Show all work! The thought process and your ability to show how and why you arrived at your answer is more important than the answer itself.
- The exam will be designed so that you can complete it without a calculator. If you find yourself yearning for a calculator, you might be doing something wrong.
- Make sure you have answered the question that you were asked. Also, ask yourself if your answer makes sense.
- If you know you made a mistake, but you can't find it, explain why you think you made a mistake and indicate where the mistake might be. This shows that you have a good understanding of the problem.
- If you write down an "=" sign, then you better be sure that the two expressions on either side are equal. Similarly, if two things are equal and it is necessary that they be equal to make your conclusion, then you better use "=."
- Don't forget to write limits where they are needed.

Related Rates

1. Suppose x and y are differentiable functions of t and are related by $y = x^2 - 1$. Find dy/dt when $x = 2$ given that $dx/dt = 3$.
2. A rock is dropped into a calm pond, causing ripples in the shape of concentric circles. The radius of the outer ripple is increasing at a rate of 12 cm/s. When the radius is 30 cm, at what rate is the total area of the outer ripple changing?
3. A spherical balloon is deflated so that its volume decreases at a constant rate of 3 in^3/s. How fast is the balloon's diameter decreasing when the radius is 2 inches?
4. A 10-foot ladder is leaning against a building. If the top of the ladder slides down the wall at a constant rate of 2 feet per second, how fast is the acute angle that the ladder makes with the ground decreasing when the top of the ladder is 5 feet from the ground? (Give your answer in radians per second.)
5. Two cars start moving from the same point. One travels south at 60 miles per hour and the other travels west at 25 miles per hour. At what rate is the distance between the cars increasing two hours later?

Linear Approximation

1. Determine the local linearization of each of the following functions at the indicated value.

   (a) $f(x) = \tan(x)$ at $x = \pi/6$.

   (b) $g(x) = \ln(e^x + e^{2x})$ at $x = 0$.

2. Suppose the local linearization of g at 2 is given by $l_2(x) = 3x - 9$. Determine $h'(2)$ if $h = f \circ g$ and $f(x) = e^{4x} - 5$.

3. Use local linearization to approximate each of the following.

   (a) $\ln(0.9)$.

   (b) $\sqrt{101}$.

Critical Numbers, Extrema, & Shape of a Graph

4. Provide an example of each of the following.

   (a) An *equation* of a function f such that f has a critical number at $x = 0$, but f does not have a local maximum or local minimum at $x = 0$.

   (b) An *equation* of a function g such that $g''(0) = 0$, but g does not have an inflection point at $x = 0$.

5. Find the critical numbers for each of the following functions.

   (a) $f(t) = 2t^3 + 3t^2 + 6t + 4$

   (b) $g(r) = \dfrac{r}{r^2 + 1}$

   (c) $h(x) = \sqrt{x}(1 - x)$

   (d) $f(\theta) = \sin^2(2\theta)$

6. Let $f(x) = \frac{6}{5}x^{5/3} - \frac{9}{2}x^{2/3}$. Find all critical numbers of f and then classify each critical number as a *local minimum*, *local maximum*, or *neither*. Sufficient work must be shown.

7. Let $f(x) = \dfrac{x^3}{3} + x^2 - 3x$.

   (a) Find the critical numbers of f.

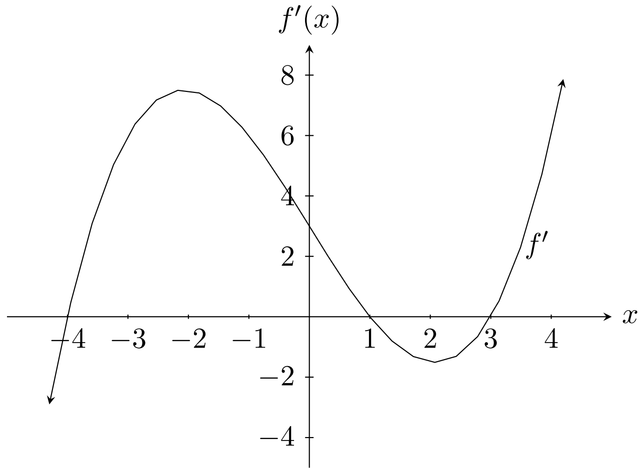
   (b) List the intervals where f is increasing.

   (c) List the intervals where f is decreasing.

   (d) Does f have any local minimums? If so, list the corresponding x-values.

   (e) Does f have any local maximums? If so, list the corresponding x-values.

8. Suppose f is a differentiable function such that the graph of f' is given below. Note this is the graph of f', NOT f.



(a) List the critical numbers of f.

(b) Find the interval(s) where f is increasing.

(c) Find the interval(s) where f is decreasing.

(d) Classify whether f has a local maximum, local minimum, or neither at each critical number.

9. A function f has a local minimum at $x = -1$ and $x = 3$ and a local maximum at $x = 2$. Sketch possible graphs for both f and f'.

10. Find the absolute maximum and minimum of $f(x) = \cos(x) - x$ on the interval $[0, 2\pi]$.

11. Find the absolute maximum and minimum of $g(x) = x^3 - 3x + 1$ on the interval $[0, 3]$.

12. Let $f(x) = \dfrac{x^5}{20} - \dfrac{x^4}{6} + \dfrac{x^3}{6} + 5x + 1$. Find the x-values of all inflection points for the graph of f.

Mean Value Theorem

13. If $f(x) = 10 - \frac{16}{x}$, then f satisfies the hypotheses of the Mean Value Theorem on the interval $[2, 8]$. Find the number c that the Mean Value Theorem guarantees exists.

14. Consider the function $f(x) = \dfrac{x}{x+2}$.

    (a) Verify that f satisfies the hypotheses of the Mean Value Theorem on the interval $[1, 4]$ and then find the number c that the Mean Value Theorem guarantees exists.

    (b) Why does f not satisfy the hypotheses of the Mean Value Theorem on the interval $[-8, 6]$?

15. A truck driver handed in a ticket at a toll booth showing that in 2 hours he had covered 158 miles on a toll road with speed limit 70 mph. The driver was cited for speeding. Use the Mean Value Theorem to explain why. State the assumptions that we have to make about the position function $p(t)$ of the truck to be able to apply the Mean Value Theorem.

L'Hôpital's Rule

Evaluate each of the following limits. If you make use of L'Hôpital's Rule, indicate where.

16. $\lim_{x\to\infty} \dfrac{1-x^3}{2x^3-5x^2+1}$

17. $\lim_{x\to-\infty} \dfrac{1-x^3}{2x^3-5x^2+1}$

18. $\lim_{x\to 0} \dfrac{1-x^3}{2x^3-5x^2+1}$

19. $\lim_{x\to 0} \dfrac{x}{e^{4x}-1}$

20. $\lim_{x\to\infty} \dfrac{x}{e^{4x}-1}$

21. $\lim_{x\to 2} \dfrac{x}{e^{4x}-1}$

22. $\lim_{x\to 0} \dfrac{\sin(3x)}{\sin(5x)}$

23. $\lim_{x\to 2} \dfrac{2e^{x-2}-x}{x^2-4}$

24. $\lim_{x\to\infty} x^2e^{-x^2}$

25. $\lim_{x\to 0} \dfrac{4x^3}{e^x}$

26. $\lim_{x\to\infty} (\ln(x))^{\frac{1}{x}}$

27. $\lim_{x\to 0^+} [\sin(x)]^{\frac{1}{x}}$

28. $\lim_{x\to 0+} \dfrac{\sin(x)}{\ln(x)}$

29. $\lim_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\sin(x)}\right)$

Curve Sketching

30. Sketch the graph of a function f that has the following properties.

(1) $f(-5)=0, f(-3)=-3, f(-2)=0$

(2) $f(-1.5)=.5, f(-.5)=1, f(1.5)=2.5$

(3) $\lim_{x\to 0} f(x)=\infty$ and $\lim_{x\to 3} f(x)=\infty$

(4) $\lim_{x\to\infty} f(x)=1$ and $\lim_{x\to-\infty} f(x)=1$

(5) $f'(-3)$ undefined

(6) $f'(1.5)=0, f'(-1.5)=0$

(7) $f'(x)>0$ on $(-3,-1.5)$, $(-1.5,0)$, $(1.5,3)$

(8) $f'(x)<0$ on $(-\infty,-3), (0,1.5)$, $(3,\infty)$

(9) $f''(x)>0$ on $(-1.5,0), (0,3)$, $(3,\infty)$

(10) $f''(x)<0$ on $(-\infty,-3)$, $(-3,-1.5)$

31. Sketch the graph of each of the following functions.

(a) $f(x)=8x^3-2x^4$

(b) $g(x)=3x^4-8x^3+6x^2+1$

(c) $f(x)=\dfrac{2(x^2-9)}{x^2-4}$

(d) $g(x)=\dfrac{-x}{(x^2-1)^2}$

(e) $h(x)=x^{5/3}-5x^{2/3}$

(f) $g(x)=x\ln(x)$

Miscellaneous

32. Determine whether the following statement is true or false.

True or **False**: If $f'(x)=g'(x)$, then $f(x)=g(x)$.

33. Find a function f that satisfies $f'(x)=\sqrt{x}$ and $f(4)=0$.