Supplemental Problems for Exam 4

General Information

Exam 4 covers material that we have covered since the first day of class up until the end of class on Wednesday, November 30. However, the exam focuses on content covered since Exam 3. In particular, here is a list of topics/concepts (since Exam 3) that you must have an understanding of or be able to do:

- Have an intuitive understanding of derivatives, based on your knowledge of rate of change, speed, velocity, and slope of tangent lines.
- Solve applied optimization problems.
- Find antiderivatives and indefinite integrals.
- Approximate the value of a definite integral, especially those involving piecewise defined functions, using left and right Riemann sums.
- Evaluate a definite integral of a continuous function using the limit or Riemann sums and right endpoints.
- Evaluate a definite integral by interpreting it as a signed area of known geometric shapes (rectangles, triangles, and circles).
- Know statements of both parts of the Fundamental Theorem of Calculus. In particular, you should know what the hypotheses are.
- Evaluate definite integrals using Part 1 of the Fundamental Theorem of Calculus.
- Compute derivative of a function that is defined in terms of an integral using Part 1 of the Fundamental Theorem of Calculus.
- Determine net change and total change using integrals.
- Evaluate integrals using substitution.
- Be able to give examples and counter examples to demonstrate different properties of functions.
- Call upon your own mental faculties to respond in flexible, thoughtful and creative ways to problems that may seem unfamiliar on first glance.

The problems that follow will provide you with an opportunity to review the relevant topics. However:

This is not a practice test!

It is possible that problems on your exam will resemble problems on this review, but you should not expect exam problems to be identical to ones found below. This review contains an abundance of problems and it is not the intention that every student will complete every problem. You should complete as many problems in each section below as you think are necessary to solidify your understanding.

In addition, you should review examples done in class, as well as your homework exercises, especially the ones on the Weekly Homework assignments.

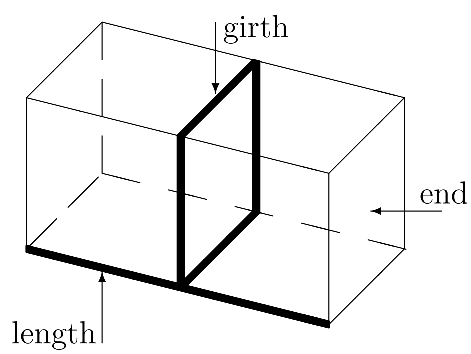
Words of Advice

Here are few things to keep in mind when taking your exam:

- Show all work! The thought process and your ability to show how and why you arrived at your answer is more important than the answer itself.
- The exam will be designed so that you can complete it without a calculator. If you find yourself yearning for a calculator, you might be doing something wrong.
- Make sure you have answered the question that you were asked. Also, ask yourself if your answer makes sense.
- If you know you made a mistake, but you can't find it, explain why you think you made a mistake and indicate where the mistake might be. This shows that you have a good understanding of the problem.
- If you write down an "=" sign, then you better be sure that the two expressions on either side are equal. Similarly, if two things are equal and it is necessary that they be equal to make your conclusion, then you better use "=."
- Don't forget to write limits where they are needed.

Applied Optimization

1. Find two positive numbers such that their product is 192 and the sum of the first and three times the second is as small as possible.
2. A farmer has 500 feet of fencing with which to enclose a pasture for grazing nuggets. The farmer only need to enclose 3 sides of the pasture since the remaining side is bounded by a river (no, nuggets can't swim). In addition, some of the nuggets don't get along with some of the other nuggets. He plans to separate the troublesome nuggets by forming 2 adjacent corrals. Determine the dimensions that would yield the maximum area for the pasture.
3. The U.S. Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 108 inches. What dimensions (length and width) will give a box with a square end the largest possible volume? *Note:* Since the box has a square end, the girth is four times the width in this case.



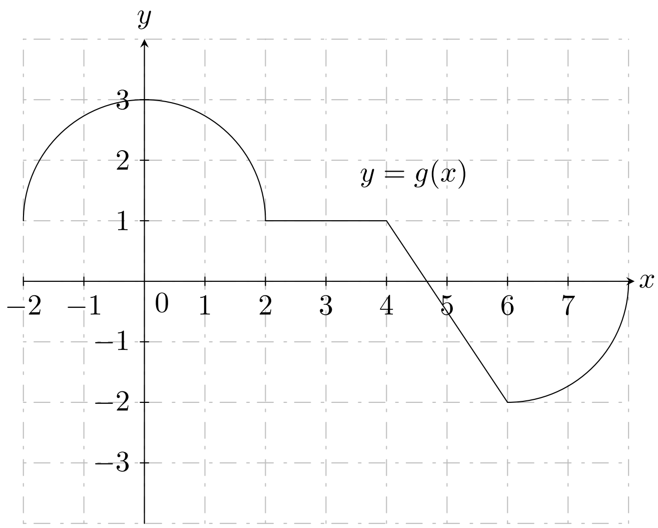
4. An open box is to be made from a 16-inch by 30-inch piece of cardboard by cutting out squares of equal size from each of the four corners and bending up the sides. What size should the squares be to obtain a box with the largest volume?

5. A rectangular piece of cardboard that is 10 inches by 15 inches is being made into a box without a top. To do so, squares are cut from each corner of the box and the remaining sides are folded up. If the box needs to be at least 1 inch deep and no more than 3 inches deep, what is the maximum possible volume of the box? what is the minimum volume?

6. A soup can in the shape of a right circular cylinder is to be made from two materials. The material for the side of the can costs \$0.015 per square inch and the material for the lids costs \$0.027 per square inch. Suppose that we desire to construct a can that has a volume of 16 cubic inches. What dimensions minimize the cost of the can?

Intuitive Definite Integral

7. Consider the graph of the function g that is given below. Assume that the graph is built from line segments, semi-circles, and quarter-circles.



Using the graph above, complete each of the following.

(a) Find $\displaystyle\int_0^4 g(x)\,dx$

(b) Find $\displaystyle\int_{-2}^8 g(x)\,dx$

8. Evaluate the following definite integrals using the graph of the function and basic area formulas.

(a) $\displaystyle\int_0^3 x+2\,dx$

(b) $\displaystyle\int_{-2}^8 |x-3|+2\,dx$

(c) $\displaystyle\int_{-3}^3 -\sqrt{9-x^2}\,dx$

(d) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 4\tan(x^5)\,dx$

Riemann Sums

9. Consider the function g given in Problem 7. Find the Riemann sum that approximates $\displaystyle\int_0^8 g(x)\,dx$ using 4 equal width subdivisions and:

(a) left end points.

(b) right end points.

10. Find the Riemann sum that approximates $\int_0^1 \sqrt{1+x^3}\ dx$ using 5 equal width subdivisions and right endpoints. Do *not* evaluate your expression.

11. The following sum is a Riemann sum that approximates the definite integral $\int_2^b f(x)\ dx$ using a partition of the interval $[2, b]$ into 4 subintervals of equal width:

$$3(\sqrt{5}+1) + 3(\sqrt{8}+1) + 3(\sqrt{11}+1) + 3(\sqrt{14}+1).$$

(a) What is b?

(b) What is $f(x)$?

12. The following sum is a Riemann sum that approximates the definite integral $\int_1^b f(x)\ dx$ using a partition of the interval $[1, b]$ into n subintervals of equal width:

$$\frac{1}{1+\frac{2}{n}} \cdot \frac{2}{n} + \frac{1}{1+\frac{4}{n}} \cdot \frac{2}{n} + \frac{1}{1+\frac{6}{n}} \cdot \frac{2}{n} + \cdots + \frac{1}{1+\frac{2n}{n}} \cdot \frac{2}{n}.$$

(a) What is b?

(b) What is $f(x)$?

13. Evaluate the following definite integral using a limit of a Riemann sum and right endpoints.

$$\int_0^1 x^2 - x\ dx$$

Fundamental Theorem of Calculus

14. Explain why the Fundamental Theorem of Calculus cannot be used to evaluate $\int_{-1}^1 \frac{1}{x^2}\ dx$.

15. Consider the function g given in Problem 7. If G is the antiderivative of g satisfying $G(0) = 42$, what is $G(4)$? *Hint:* Consider Problem 7(a).

16. Complete the following.

(a) Let $A(x) = \int_0^x t^2 - t\ dt$. Find $A'(x)$.

(b) Let $f(x) = \int_0^x \sqrt[3]{t^2+1}\ dt$. Find $f'(x)$.

(c) Let $G(x) = \int_0^{x^2} t^3 \sin(t)\ dt$. Find $G'(x)$.

(d) Let $C(x) = \int_x^{x^3} \cos(\cos(t))\ dt$. Find $C'(x)$.

17. Let $A(x) = \int_0^x \sin^2(t)\ dt$. Determine where A attains its absolute maximum value on $[0, \pi]$.

Indefinite and Definite Integrals

Compute each of the following integrals.

18. $\int 5 \, dx$

19. $\int 0 \, dx$

20. $\int 2x^3 + x^2 - 5x + 5 \, dx$

21. $\int -2\sqrt{x} \, dx$

22. $\int \frac{x+1}{\sqrt{x}} \, dx$

23. $\int \frac{1}{x^3} \, dx$

24. $\int \frac{x+5}{x^2} \, dx$

25. $\int \frac{\sin(x)}{\cos^2(x)} \, dx$

26. $\int_0^1 x^2 \, dx$

27. $\int_{-1}^1 x^4 - \frac{1}{2}x^3 + \frac{1}{4}x - 2 \, dx$

28. $\int_0^\pi \sin(x) \, dx$

29. $\int_0^\pi \cos(2x) \, dx$

30. $\int_0^{\ln(2)} e^{x/3} \, dx$

31. $\int_1^{e^2} \frac{x+1}{x^2} \, dx$

32. $\int_1^2 \frac{x^3 - 2\sqrt{x}}{x} \, dx$

33. $\int_0^{1/2} \frac{4}{\sqrt{1-x^2}} \, dx$

34. $\int (3x-1)^2 \, dx$

35. $\int (3x-1)^{99} \, dx$

36. $\int 5x^2\sqrt{x^3-2} \, dx$

37. $\int_0^2 xe^{x^2} \, dx$

38. $\int \sin^2(x)\cos(x) \, dx$

39. $\int_0^1 \frac{x}{x^2+1} \, dx$

40. $\int x^2 \sec^2(x^3) \, dx$

41. $\int \frac{x}{x^4+1} \, dx$

42. $\int x\sqrt{x-1} \, dx$

43. $\int \frac{e^x}{e^x+1} \, dx$

44. $\int \frac{1}{\sqrt{1-9x^2}} \, dx$

Miscellaneous

45. Determine whether each of the following statements is true or false. Circle the correct answer.

    (a) **True** or **False**: $\int_a^b f(x)g(x) \, dx = \int_a^b f(x) \, dx \cdot \int_a^b g(x) \, dx$

    (b) **True** or **False**: $\int_a^b f(x) + g(x) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$

    (c) **True** or **False**: If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx$.

    (d) **True** or **False**: If $f'(x) = g'(x)$, then $f(x) = g(x)$.

(e) **True** or **False**: The formula $\displaystyle\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$ works for *all* values of n.

46. Find a function f that satisfies $f'(x) = \sqrt{x}$ and $f(4) = 0$.

47. Find f that satisfies $f''(x) = x^2 + 4, f'(3) = 1$, and $f(1) = 6$.

48. Find the antiderivative F satisfying $F(2\pi) = 2$ of the following function.

$$f(x) = \begin{cases} \sin(x), & x < 2\pi \\ x - 2\pi, & x \geq 2\pi \end{cases}$$

49. Use basic properties of integrals to evaluate the following.

(a) $\displaystyle\int_1^{11} f(x) \, dx$ if $\displaystyle\int_0^1 f(x) \, dx = -7$ and $\displaystyle\int_0^{11} f(x) \, dx = 29$

(b) $\displaystyle\int_0^4 5f(x) + \sqrt{x} \, dx$ if $\displaystyle\int_0^{10} f(x) \, dx = 8$ and $\displaystyle\int_{10}^4 f(x) \, dx = -3$

50. Find a positive value of a such that $\displaystyle\int_a^{2a} \frac{3}{4} x(x^2 - a^2)^2 \, dx = 1$.

51. A zombie moves in a straight line with velocity $v(t) = -t + 4$ mph after t hours of his start. How far is he from his original position after 6 hours?

52. A bungie jumper jumps off a bridge. Her downward velocity in feet per second, after t seconds of the fall, is $v(t) = 160(1 - e^{-t/5})$. This function is good for the first 5 seconds, but after $t = 5$ the bungie cord slows her fall. How far did she fall in those 5 seconds?