The Fundamental Theorem of Calculus

Goal

In this section, we will introduce the first Fundamental Theorem of Calculus, which is the crowning achievement of calculus. In particular, the FTC tells us how net signed area under a curve and antiderivatives are related.

Background material

Recall that the definite integral is defined as

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x,$$

as long as this limit exists for all possible choices of c_i. The definite integral is a number that represents the net signed area under a curve over an interval.

Also, recall that the collection of all antiderivatives of a function f is given by the indefinite integral

$$\int f(x)\ dx = F(x) + C,$$

where F is an antiderivative of f. The indefinite integral is a family of functions.

Of course, the notation between these two concepts is similar, but how are they related?

The First Fundamental Theorem of Calculus

The following theorem is one of the punchlines of calculus.

Theorem 1 (FTC1)**.** Let f' be a integrable function over the interval $[a, b]$ (i.e., the limit of Riemann sums exists). Then

$$\boxed{\int_a^b f'(x)\ dx = f(b) - f(a).}$$

Proof. Let $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be an equal width subdivision of $[a, b]$. We see that

$$\begin{aligned} f(b) - f(a) &= f(x_n) - f(x_0) \\ &= f(x_n) - f(x_{n-1}) + f(x_{n-1}) - f(x_{n-2}) + f(x_{n-2}) - \cdots - f(x_1) + f(x_1) - f(x_0) \\ &= \left(\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} + \cdots + \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right) \Delta x \end{aligned}$$

But by the Mean Value Theorem, on each $[x_{i-1}, x_i]$, there exists c_i such that

$$f'(c_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}.$$

This implies that

$$\begin{aligned} f(b) - f(a) &= (f'(c_n) + \cdots + f'(c_1))\Delta x \\ &= \sum_{i=1}^{n} f'(c_i)\Delta x. \end{aligned}$$

Lastly, taking limit as $n \to \infty$, we obtain

$$f(b) - f(a) = \int_a^b f'(x)\ dx,$$

as desired. □

Note 2. If we are given a function f that is integrable over $[a, b]$ and we know that F is an antiderivative of f, then

$$\boxed{\int_a^b f(x)\ dx = F(b) - F(a).}$$

Important Note 3. If f is continuous on $[a, b]$, we do not have to use the cumbersome limit of Riemann sums, but rather we can subtract the values of the antiderivative of f at the endpoints. This is amazing! If f is *not* continuous on $[a, b]$, then all bets are off.

Examples

Let's crank through some examples.

Example 4. Compute each of the following definite integrals.

(a) $\displaystyle\int_0^1 x^2\ dx$

(b) $\displaystyle\int_{-1}^1 x^4 - \frac{1}{2}x^3 + \frac{1}{4}x - 2\ dx$

(c) $\displaystyle\int_0^\pi \sin(x)\ dx$

(d) $\displaystyle\int_0^{\pi} \cos(2x)\ dx$

(e) $\displaystyle\int_1^{2} \frac{x^3 - 2\sqrt{x}}{x}\ dx$

(f) $\displaystyle\int_1^{e} \frac{x+1}{x}\ dx$

(g) $\displaystyle\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\ dx$

Example 5. Explain why the Fundamental Theorem of Calculus cannot be used to evaluate the following integral.

$$\int_{-1}^{1} \frac{1}{x^2}\ dx$$

The Second Fundamental Theorem of Calculus

Definition 6. Let f be a continuous function. Then the *accumulation function* A of f at a is defined via

$$\boxed{A(x) = \int_a^x f(t)\ dt.}$$

Notice that the variable of A is the upper limit of the integral. Loosely speaking, $A(x)$ is the amount of paint that we need to paint under the graph of f from a to x (where a is fixed, but x is free to move around).

The Picture:

Example 7. Let $A(x) = \displaystyle\int_0^x t^2\ dt$.

(a) Find $A(1)$.

(b) Determine a formula for $A(x)$ that does not involve an integral.

(c) Find A'. Any observations?

Note 8. Our ability to successfully answer part (b) in the previous example hinged on our ability to find an antiderivative of f. This isn't always possible! However, it turns out that we don't need to be able to accomplish part (b) to be able to pull off part (c).

Theorem 9 (FTC2)**.** Let f be a continuous function. Then

$$\boxed{\frac{d}{dx}\left[\int_a^x f(t)\ dt\right] = f(x).}$$

Proof. Let

$$A(x) = \int_a^x f(t)\ dt,$$

so that A is an accumulation function. Since f is continuous, f is integrable over $[a, b]$. By FTC1, we have

$$A(x) = \int_a^x f(t)\ dt = F(x) - F(a),$$

where F is an antiderivative of f. This implies that

$$A'(x) = \frac{d}{dx}\left[\int_a^x f(t)\ dt\right] = F'(x) - 0 = f(x),$$

as desired. $\square$

Note 10. There are a few observations that we should make.

1. First, note that one consequence of our proof is that A is an antiderivative of f.

2. It is also the case that when h is small, we have

$$\frac{A(x+h) - A(x)}{h} \approx f(x) \iff A(x+h) - A(x) \approx f(x) \cdot h.$$

   Said another way, the rate of change of the accumulation function at x is the height of f at x. Here's a picture:

3. If the upper limit is something else besides x, we can utilize the chain rule in conjunction with FTC2 to determine the derivative. In particular, we have

$$\frac{d}{dx}\left[\int_a^u f(t)\ dt\right] = f(u) \cdot \frac{du}{dx},$$

   where u is some function of x and f is continuous. That is, replace t with u and remember to multiply by the derivative of u.

Examples

Let's try some examples.

Example 11. Find the corresponding derivatives.

(a) Let $A(x) = \int_0^x t^2\, dt$. Find A' and compare with earlier example.

(b) Let $f(x) = \int_0^x \sqrt[3]{t^2+1}\, dt$. Find f'.

(c) Let $G(x) = \int_0^{x^2} t^3 \sin(t)\, dt$. Find G'.

(d) Let $C(x) = \int_x^{x^3} \cos(\cos(t))\, dt$. Find C'.

Example 12. Let $A(x) = \int_0^x \sin^2(t)\, dt$. Determine where A attains a maximum value on the interval $[0, \pi]$.

