

# Integration by Substitution

## Motivation and Background

Currently, we do *not* have a technique for integrating most products, quotients, and compositions. Here are a couple that we can integrate:

$$\int \frac{x^2 + x}{\sqrt{x}} dx, \quad \int \sec(x) \tan(x) dx$$

And here are some that we cannot currently integrate (unless you happen to see what the appropriate antiderivative is):

$$\int x\sqrt{x^2 + 1} dx, \quad \int \sin(x) \cos(x) dx, \quad \int \frac{x}{x^2 + 1} dx$$

To integrate functions like above, we will utilize a technique called *substitution*, which involves the use of dummy variable.

**Important Note 1.** Substitution is a technique that only works in special circumstances, which should become apparent after a little practice.

## Differentials

Before diving into substitution, we need to have a quick discussion about differentials.

**Definition 2.** If  $y = f(x)$ , then we define the *differential of  $y$*  to be

$$dy = f'(x)dx.$$

**Note 3.** Here are a few comments about differentials.

1. Recall that if  $y = f(x)$ , then the corresponding derivative can be written as  $\frac{dy}{dx} = f'(x)$ . So, from a “symbol-shoving” perspective, it looks like differentials are the result of just moving the  $dx$  over to the other side of the previous equation.
2. If  $y = f(x)$ , then  $dy$  measures the change in  $y$  for the tangent line at a given point.
3. If we write  $u = f(x)$ , then the differential of  $u$  is  $du = f'(x)dx$ .

**Example 4.** If  $u = x^2 + 1$ , find  $du$ .

**Important Note 5.** If confronted with an integral of a product, quotient, or composition and you cannot integrate it straight away, then substitution may work. In most (but definitely not all) situations, you will pick  $u$  to be the inside of the more complicated part.

## Examples

We'll introduce the technique of substitution by way of examples.

**Example 6.** Compute each of the following integrals.

1.  $\int (3x - 1)^{99} dx$

2.  $\int 5x^2 \sqrt{x^3 - 2} dx$

$$3. \int x e^{x^2} dx$$

$$4. \int \sin^2(x) \cos(x) dx$$

$$5. \int \frac{x}{x^2 + 1} dx$$

$$6. \int \frac{x^2 + 1}{x} dx$$

$$7. \int x^2 \sec^2(x^3) dx$$

$$8. \int_1^e \frac{\ln(x)}{x} dx$$

$$9. \int_0^{1/2} \frac{\arcsin(x)}{\sqrt{1-x^2}} dx$$

$$10. \int_0^1 \frac{e^x}{e^{2x} + 1} dx$$

$$11. \int \frac{x}{x^4 + 1} dx$$

$$12. \int x\sqrt{x-1} dx$$