Chapter 7

Functions

7.1 Introduction to Functions

The concept of function is one of the most important and fundamental ones in the field of mathematics. Functions are used in all branches of mathematics to model diverse situations and pull together ideas that at first seem unrelated. Functions are as vital as numbers.

Undoubtably, you have encountered the concept of function in your prior mathematical experience. In this section, we will introduce the concept of function as a special type of relation. As you shall see, this agrees with any previous definition of function that you may have learned.

Up until this point, you've probably only encountered functions as an algebraic rule, e.g., $f(x) = x^2 - 1$, for transforming one real number into another. However, we can study functions in a much broader context. Loosely speaking, the basic building blocks of a function are a first set and a second sets, say X and Y, respectively, and a "correspondence" that assigns each element of X to exactly one element of Y. Let's take a look at the actual definition.

Definition 7.1. Let X and Y be two nonempty sets. A **function** from set X to set Y, denoted $f : X \to Y$, is a relation (i.e., subset of $X \times Y$) such that:

(a) For each $x \in X$, there exists $y \in Y$ such that $(x, y) \in f$, and

(b) If $(x, y_1), (x, y_2) \in f$, then $y_1 = y_2$.

Note that if $(x, y) \in f$, we usually write $y = f(x)$ and say that "f maps x to y."

Part (a) of Definition 7.1 says that every element of X appears in the first coordinate of an ordered pair in the relation. Part (b) says that each element of X only appears once in the first coordinate of an ordered pair in the relation. It is important to note that there are no restrictions on whether an element of Y ever appears in the second coordinate. Furthermore, if an element of Y appears in the second coordinate, it may appear again in a different ordered pair.

Definition 7.2. The set X from Definition 7.1 is called the **domain** of f and is denoted by $\mathrm{Dom}(f)$. The set Y is called the **codomain** of f and is denoted by $\mathrm{Codom}(f)$. The set

$$\mathrm{Rng}(f) = \{y \in Y \mid \text{there exists } x \text{ such that } y = f(x)\}$$

is called the **range** of f or the **image of** X under f. If f is a function and $(x,y) \in f$, then we may refer to x as the **input** of f and y as the **output** of f.

It follows immediately from the definition that $\mathrm{Rng}(f) \subseteq \mathrm{Codom}(f)$. However, it is possible that the range of f is a proper subset of of the codomain.

Exercise 7.3. Let $X = \{\circ, \square, \triangle, ☺\}$ and $Y = \{a,b,c,d,e\}$. Determine whether each of the following represent functions. Explain. If the relation is a function, determine the domain, codomain, and range.

(a) $f : X \to Y$ defined via $f = \{(\circ,a),(\square,b),(\triangle,c),(☺,d)\}$.

(b) $g : X \to Y$ defined via $g = \{(\circ,a),(\square,b),(\triangle,c),(☺,c)\}$.

(c) $h : X \to Y$ defined via $h = \{(\circ,a),(\square,b),(\triangle,c),(\circ,d)\}$.

(d) $k : X \to Y$ defined via $k = \{(\circ,a),(\square,b),(\triangle,c),(☺,d),(\square,e)\}$.

(e) $l : X \to Y$ defined via $l = \{(\circ,e),(\square,e),(\triangle,e),(☺,e)\}$.

(f) $m : X \to Y$ defined via $m = \{(\circ,a),(\triangle,b),(☺,c)\}$.

(g) $\mathrm{happy} : Y \to X$ defined via $\mathrm{happy}(y) = ☺$ for all $y \in Y$.

(h) $\mathrm{id} : X \to X$ defined via $\mathrm{id}(x) = x$ for all $x \in X$.

(i) $\mathrm{nugget} : X \to X$ defined via

$$\mathrm{nugget}(x) = \begin{cases} x, & \text{if } x \text{ is a geometric shape,} \\ \square, & \text{otherwise.} \end{cases}$$

One useful representation of functions on finite sets is via **bubble diagrams**. To draw a bubble diagram for a function $f : X \to Y$, draw one circle (i.e, a "bubble") for each of X and Y and for each element of each set, put a dot in the corresponding set. Typically, we draw X on the left and Y on the right. Next, draw an arrow from $x \in X$ to $y \in Y$ if $f(x) = y$ (i.e., $(x,y) \in f$). Note that we can draw bubble diagrams even if f is not a function.

Example 7.4. Figure 7.1 depicts a bubble diagram for a function from domain $X = \{a,b,c,d\}$ to codomain $Y = \{1,2,3,4\}$. In this case, the range is equal to $\{1,2,4\}$.

Exercise 7.5. For each of the relations in Exercise 7.3 draw the corresponding bubble diagram.

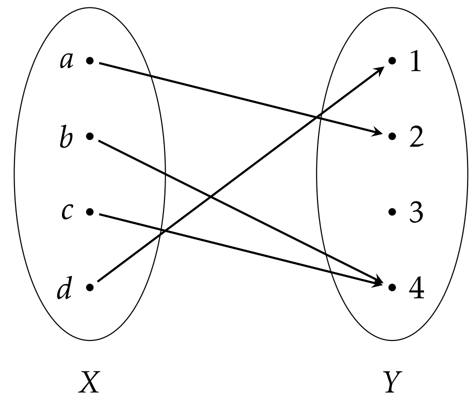

Figure 7.1: An example of a bubble diagram for a function.

Problem 7.6. What properties does a bubble diagram have to have in order to represent a function?

Exercise 7.7. Provide an example of each of the following. You may draw a bubble diagram, write down a list of ordered pairs, or a write a formula (as long as the domain and codomain are clear).

(a) A function f from a set with 4 elements to a set with 3 elements such that $\mathrm{Rng}(f) = \mathrm{Codom}(f)$.

(b) A function g from a set with 4 elements to a set with 3 elements such that $\mathrm{Rng}(g)$ is strictly smaller than $\mathrm{Codom}(g)$.

Problem 7.8. Let $f : X \to Y$ be a function and suppose that X and Y are finite sets with n and m elements, respectively, such that $n < m$. Is it possible for $\mathrm{Rng}(f) = \mathrm{Codom}(f)$? Explain.

Problem 7.9. In high school I am sure that you were told that a graph represents a function if it passes the **vertical line test**. Using our terminology of ordered pairs, explain why this works.

Definition 7.10. Two functions are equal if they have the same domain, same codomain, and the same set of ordered pairs in the relation. That is, if $f : X \to Y$ and $g : X \to Y$ are functions, then $f = g$ iff $f(x) = g(x)$ for all $x \in X$.

If two functions are defined by the same algebraic formula, but have different domains, then they are *not* equal. For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x^2$ is not equal to the function $g : \mathbb{N} \to \mathbb{N}$ defined via $g(x) = x^2$.

Definition 7.11. Let $f : X \to Y$ be a function.

(a) The function f is said to be **one-to-one** (or **injective**) if for all $y \in \mathrm{Rng}(f)$, there is a unique $x \in X$ such that $y = f(x)$.

(b) The function f is said to be **onto** (or **surjective**) if for all $y \in Y$, there exists $x \in X$ such that $y = f(x)$.

(c) If f is both one-to-one and onto, we say that f is a **one-to-one correspondence** (or a **bijection**).

Remark 7.12. Let $f : X \to Y$ be a function. To prove that f is one-to-one, start by assuming that $f(x_1) = f(x_2)$ and then work to show that $x_1 = x_2$. That is, a function f is one-to-one iff for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. To show that f is onto, you should start with an arbitrary $y \in Y$ and then work to show that there exists $x \in X$ such that $y = f(x)$.

Exercise 7.13. Provide an example of each of the following. You may draw a bubble diagram, write down a list of ordered pairs, or write a formula (as long as the domain and codomain are clear). Assume that X and Y are finite sets.

(a) A function $f : X \to Y$ that is one-to-one but not onto.

(b) A function $f : X \to Y$ that is onto but not one-to-one.

(c) A function $f : X \to Y$ that is both one-to-one and onto.

(d) A function $f : X \to Y$ that is neither one-to-one nor onto.

Problem 7.14. Perhaps you've heard of the **horizontal line test** (i.e., every horizontal line hits the graph of $f : \mathbb{R} \to \mathbb{R}$ at most once). What is the horizontal line test testing for?

Exercise 7.15. Provide an example of each of the following. You may either draw a graph or write down a formula. Make sure you have the correct domain.

(a) A function $f : \mathbb{R} \to \mathbb{R}$ that is one-to-one but not onto.

(b) A function $f : \mathbb{R} \to \mathbb{R}$ that is onto but not one-to-one.

(c) A function $f : \mathbb{R} \to \mathbb{R}$ that is both one-to-one and onto.

(d) A function $f : \mathbb{R} \to \mathbb{R}$ that is neither one-to-one nor onto.

Exercise 7.16. Determine which of the following functions are one-to-one, onto, both, or neither. In each case, you should provide proofs and counterexamples as appropriate.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x^2$

(b) $g : \mathbb{R} \to [0, \infty)$ defined via $g(x) = x^2$

(c) $h : \mathbb{R} \to \mathbb{R}$ defined via $h(x) = x^3$

(d) $k : \mathbb{R} \to \mathbb{R}$ defined via $k(x) = x^3 - x$

(e) $l : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined via $l(x_1, x_2) = x_1^2 + x_2^2$

(f) $N : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ defined via $N(n) = (n, n)$

Exercise 7.17. Let A and B be sets and let $S \subseteq A \times B$. Define $\pi_1 : S \to A$ and $\pi_2 : S \to B$ via $\pi_1(a,b) = a$ and $\pi_2(a,b) = b$. We call π_1 (respectively, π_2) the **projections** of S onto A (respectively, B).

(a) Provide examples to show that π_1 does not need to be one-to-one or onto.

(b) Suppose that S is a function (recall that a function is a set of ordered pairs, so this makes sense). Is π_1 one-to-one? Is π_1 onto? How about π_2?

7.2 Compositions and Inverses

Definition 7.18. If $f : X \to Y$ and $g : Y \to Z$ are functions, then a new function $g \circ f : X \to Z$ can be defined by $(g \circ f)(x) = g(f(x))$ for all $x \in \mathrm{Dom}(f)$.

It is important to notice that the function on the right is the one that "goes first."

Exercise 7.19. In each case, give examples of finite sets X, Y, and Z, and functions $f : X \to Y$ and $g : Y \to Z$ that satisfy the given conditions. Drawing bubble diagrams is sufficient.

(a) f is onto, but $g \circ f$ is not onto.

(b) g is onto, but $g \circ f$ is not onto.

(c) f is one-to-one, but $g \circ f$ is not one-to-one.

(d) g is one-to-one, but $g \circ f$ is not.

Theorem 7.20. If $f : X \to Y$ and $g : Y \to Z$ are both functions that are onto, then $g \circ f$ is also onto.

Theorem 7.21. If $f : X \to Y$ and $g : Y \to Z$ are both functions that are one-to-one, then $g \circ f$ is also one-to-one.

Corollary 7.22. If $f : X \to Y$ and $g : Y \to Z$ are both one-to-one correspondences, then $g \circ f$ is also a one-to-one correspondence.

Problem 7.23. Assume that $f : X \to Y$ and $g : Y \to Z$ are both functions. For each of the following statements, if the statement is true, then prove it. If the statement is false, provide a counterexample.

(a) If $g \circ f$ is one-to-one, then f is one-to-one.

(b) If $g \circ f$ is one-to-one, then g is one-to-one.

(c) If $g \circ f$ is onto, then f is onto.

(d) If $g \circ f$ is onto, then g is onto.

Definition 7.24. Let $f : X \to Y$ be a function. The relation f^{-1}, called f **inverse**, is defined via

$$f^{-1} = \{(f(x), x) \in Y \times X \mid x \in X\}.$$

Notice that we called f^{-1} a relation and not a function. In some circumstances f^{-1} will be a function and sometimes it will not be.

Exercise 7.25. Provide an example of a function $f : X \to Y$ such that f^{-1} is *not* a function. A bubble diagram is sufficient.

Exercise 7.26. Provide an example of a function $f : X \to Y$ such that f^{-1} is a function. A bubble diagram is sufficient.

Theorem 7.27. Let $f : X \to Y$ be a function. Then f^{-1} is a function iff f is ____________.

Theorem 7.28. Let $f : X \to Y$ be a function and suppose that f^{-1} is a function. Then

(a) $(f \circ f^{-1})(x) = x$ for all $x \in Y$, and

(b) $(f^{-1} \circ f)(x) = x$ for all $x \in X$.

Theorem 7.29. Let $f : X \to Y$ and $g : Y \to X$ be functions such that f is a one-to-one correspondence. If $(f \circ g)(x) = x$ for all $x \in Y$ and $(g \circ f)(x) = x$ for all $x \in X$, then $g = f^{-1}$.

The upshot of the previous two theorems is that if f^{-1} is a function, then it is the only one satisfying the two-sided "undoing" property exhibited in Theorem 7.28. The next theorem can be considered to be the converse of Theorem 7.29.

Theorem 7.30. Let $f : X \to Y$ and $g : Y \to X$ be functions satisfying $(f \circ g)(x) = x$ for all $x \in Y$ and $(g \circ f)(x) = x$ for all $x \in X$. Then f is a one-to-one correspondence.

Theorem 7.31. Let $f : X \to Y$ and $g : Y \to Z$ be functions. If f and g are both one-to-one correspondences, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.