MAT 411: Introduction to Abstract Algebra
Exam 2 (Take-Home Portion)

Your Name:

Names of Any Collaborators:

Instructions

This portion of Exam 2 is worth a total of 37 points and is due at the beginning of class on **Monday, October 31**. Your total combined score on the in-class portion and take-home portion is worth 15% of your overall grade.

I expect your solutions to be *well-written, neat, and organized.* Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem 1.41, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

1. Let G be a group and let $H \leq G$. Define the relation $\sim$ on G via
$$a \sim b \text{ iff } a^{-1}b \in H.$$
   (a) (4 points) Prove that $\sim$ is an equivalence relation on G.
   (b) (2 points) If $[a]$ represents the equivalence class containing a under $\sim$, prove that $[a] = aH$, where $aH := \{ah \mid h \in H\}$.
   (c) (4 points) Define $\phi : H \to aH$ via $\phi(h) = ah$. Prove that ϕ is one-to-one and onto.*
   (d) (4 points) Prove that if G is a finite group, then $|H|$ divides $|G|$.
   (e) (2 points) Prove that if G is a finite group and $a \in G$, then $|a|$ divides $|G|$.

   You may use the results above (even if you were not able to prove them) on the rest of the exam.

2. (4 points each) Suppose G is a group of order pq, where p and q are distinct primes (i.e., $p \neq q$).
   (a) Prove that if G is cyclic, then G has both an element of order p and an element of order q.
   (b) Prove that if G is any group of order pq (not necessarily cyclic), then G has either an element of order p or an element of order q.†

3. (4 points each) Prove any **two** of the following theorems.

   Theorem 1. Suppose $(G_1, *)$ and $(G_2, \circ)$ are groups and the function $\phi : G_1 \to G_2$ satisfies the homomorphic property. If H and K are subgroups of G_1 with $H \leq K$, then $\phi(H) \leq \phi(K)$.

   Theorem 2. Suppose $(G_1, *)$ and $(G_2, \circ)$ are groups and the function $\phi : G_1 \to G_2$ satisfies the homomorphic property. If $g \in G_1$ such that g has finite order, then $|\phi(g)|$ divides $|g|$.

   Theorem 3. Suppose $(G, *)$ is a group and define $\phi : G \to G$ via $\phi(g) = g^2$ for all $g \in G$. Then ϕ satisfies the homomorphic property iff G is abelian.

   Theorem 4. Suppose $(G, *)$ is a finite cyclic group such that $|G| = n$. Then G has no proper nontrivial subgroups iff n is prime.

4. (5 points) Consider the following five groups of order 8: $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, D_4, and Q_8. It turns out that none of these groups are isomorphic to each other. In fact, you proved this on the take-home portion of Exam 1, but we used isomorphic versions for a couple of them (e.g., $L_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $S_2 \times R_4 \cong \mathbb{Z}_4 \times \mathbb{Z}_2$). Using the pictures of the five group table quilts that I provided on the course webpage, determine which group corresponds to which quilt. You may assume that there is a one-to-one correspondence between groups and quilts. Be sure to provide sufficient justification. You may need to rotate some of the quilts to justify your answers.

*The function ϕ is not intended to satisfy the homomorphic property. In fact, it doesn't.

†Recall that in mathematics, "or" is inclusive unless specified otherwise. So, this statement allows for both an element of order p and an element of order q. It turns out that G must have both an element of order p and an element of order q, but you don't need to prove this.

