MAT 411: Introduction to Abstract Algebra
Exam 3 (Take-Home Portion)

Your Name:

Names of Any Collaborators:

Instructions

This portion of Exam 3 is worth a total of 24 points and is due by **Tuesday, December 5** by 8PM. Your total combined score on the in-class portion and take-home portion is worth 18% of your overall grade.

I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem 1.41, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

1. (4 points each) Complete **two** of the following.

   (a) Suppose G is a group and let $H \leq G$. Prove that $H \trianglelefteq G$ if and only if $aHa^{-1} \subseteq H$ for all $a \in G$, where $aHa^{-1} = \{aha^{-1} \mid h \in H\}$.*

   (b) Prove that the group $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic if and only if m and n are relatively prime.†

   (c) Suppose $G_1, G_2, \ldots, G_n$ are groups and let $(g_1, g_2, \ldots, g_n) \in \prod_{i=1}^n G_i$. Prove that if each $|g_i| < \infty$, then $|(g_1, g_2, \ldots, g_n)| = \operatorname{lcm}(|g_1|, |g_2|, \ldots, |g_n|)$.‡

   (d) Let G be a group and let $H \leq G$. Prove that left coset multiplication is well-defined if and only if $H \trianglelefteq G$.§

   (e) Let G be a group. Prove that if $G/Z(G)$ is cyclic, then G is abelian.¶

2. (2 points each) The goal of this problem is to prove the First Isomorphism Theorem (Theorem 9.20). Let G_1 and G_2 be groups and suppose $\phi : G_1 \to G_2$ is a homomorphism. Define $\psi : G_1/\ker(\phi) \to \phi(G_1)$ via $\psi(g\ker(\phi)) = \phi(g)$.

   (a) Prove that ψ is well-defined. That is, prove that if $g_1 \ker(\phi) = g_2 \ker(\phi)$, then $\psi(g_1 \ker(\phi)) = \psi(g_2 \ker(\phi))$.

   (b) Prove that ψ is a homomorphism.

   (c) Prove that ψ is onto.

   (d) Prove that ψ is one-to-one.‖

   (e) Conclude that $G_1/\ker(\phi) \cong \phi(G_1)$.

3. (3 points each) Complete **two** of the following.

   (a) Prove that if $|G| = pq$, where p and q are primes (not necessarily distinct), then either $Z(G) = \{e\}$ or G is abelian.**

   (b) Let H be a normal subgroup of a group G. Prove that if $g \in G$, then the order of gH (in G/H) divides the order of g (in G).

   (c) Use the First Isomorphism Theorem to prove that $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}_6$.

   (d) Use the First Isomorphism Theorem to prove that $(\mathbb{Z}_4 \times \mathbb{Z}_2)/(\{0\} \times \mathbb{Z}_2) \cong \mathbb{Z}_4$.

*This is a slightly weaker version of Theorem 7.33. The set aHa^{-1} is called the **conjugate** of H by a. It's not too hard to show that if $gHg^{-1} \subseteq H$ for all $g \in G$, then we actually have $gHg^{-1} = H$ for all $g \in G$, but you do not need to prove this.

†This is Theorem 8.10.

‡This is Theorem 8.16.

§This is Theorem 8.29. *Hint:* The definition of left coset multiplication tells us that $(aH)(bH) = abH$ and $(cH)(dH) = cdH$. If $aH = cH$ and $bH = dH$, then left coset multiplication is well-defined if and only if $cdH = abH$.

¶Recall that $Z(G)$ denotes the center of G (see Theorem 5.63). Note that since the elements of $Z(G)$ commute with all the elements of G, the left and right cosets of $Z(G)$ will be equal, and hence $Z(G)$ is normal in G.

‖One approach is to make use of Theorem 9.13, which you also encountered on the take-home portion of Exam 2.

***Hint:* Use Problem 1(e), which you may use even if you did not complete that problem.

