Chapter 4

An Introduction to Subgroups and Isomorphisms

In this chapter, we'll continue to utilize our intuitive definition of a group. That is, a group G is a set of actions that satisfies the following rules.

Rule 1. There is a predefined list of actions that never changes.

Rule 2. Every action is reversible.

Rule 3. Every action is deterministic.

Rule 4. Any sequence of consecutive actions is also an action.

In the previous chapter, we constructed lots of Cayley diagrams for various groups. To construct a Cayley diagram for a group G, we need to first identify a set of generators, say S. Recall that our choice of generators is important as changing the generators can result in a different Cayley diagram.

In the Cayley diagram for G using S, all the actions of G are represented by the vertices of the graph. Each vertex corresponds to a unique action. This does not imply that there is a unique way to obtain a given action from the generators. Each of the generators determines an arrow type in the diagram. One way to distinguish the different arrow types is by using different colors. An arrow of a particular color always represents the same generator.

One of the vertices in the diagram is labeled by the do-nothing action, often denoted by e. Each of the other vertices are labeled by words that correspond to following arrows (forwards or backwards) from e to a given vertex. There may be many ways to do this as each sequence of arrows corresponds to a unique word. So, a vertex could be potentially labeled by many words. Also, one potentially confusing item is that we read our words from right to left. That is, the first arrow we follow out of e is the rightmost generator in the word.

4.1 Subgroups

Exercise 4.1. Recall the definition of "subset." What do you think "subgroup" means? Try to come up with a potential definition. Try not to read any further before doing this.

Before continuing, gather up the following Cayley diagrams:

- $\text{Spin}_{1\times 2}$. There are 3 of these. I drew one for you in Chapter 3 and you discovered two more in Exercise 3.6.
- S_2. See Exercise 3.9.
- R_4. See Exercise 3.10.
- D_3. There are two of these. See exercises 3.11 and 3.14.
- D_4. See Exercise 3.12.

Exercise 4.2. Examine your Cayley diagrams for D_4 and R_4 and make some observations. How are they similar and how are they different? Can you reconcile the similarities and differences by thinking about the actions of each group?

Hopefully, one of the things you noticed in the previous exercise is that we can "see" R_4 inside of D_4 (and hopefully you didn't just read that before completing the exercise). You may have used different colors in each case and maybe even labeled the vertices with different words, but the overall structure of R_4 is there nonetheless.

Exercise 4.3. If you just pay attention to the configuration of arrows, it appears that there are two copies of the Cayley diagram for R_4 in the Cayley diagram for D_4. Isolate these two copies by ignoring the edges that correspond to the generator s. Paying close attention to the words that label the vertices from the original Cayley diagram for D_4, are either of these groups in their own right?

Recall that the do-nothing action must always be one of the actions included in a group. If this didn't occur to you when doing the previous exercise, you might want to go back and rethink your answer. Just like in the previous exercise, we can often "see" smaller groups living inside larger groups. These smaller groups are called **subgroups**.

Intuitive Definition 4.4. Let G be a group of actions and let $H \subseteq G$. We say that H is a **subgroup** if and only if H is a group in its own right. In this case, we write $H \leq G$.

In light of Exercise 4.3, we would write $R_4 \leq D_4$. The second sub-diagram of D_4 that resembles R_4 cannot be a subgroup because it does not contain the do-nothing action. However, since it looks a lot like R_4, we call it a **clone** of R_4. For convenience, we may also say that a subgroup is a clone of itself.

The next theorem* tells us that if we already have a subset of a group, we only need to check two of our rules instead of four.

*Perhaps we should call this an "Intuitive Theorem" since we are using an intuitive definition of a group.

Exercise 4.5. Let G be a group of actions and let $H \subseteq G$. If we wanted to determine whether H is a subgroup of G or not, can we skip checking any of the four rules? Which rules must we verify?

There are a couple subgroups that every group has.

Theorem 4.6. Let G be a group of actions and suppose that e is the do-nothing action. Then $\{e\} \leq G$.

Exercise 4.7. Let G be a group and suppose that e is the do-nothing action. What does the Cayley diagram for the subgroup $\{e\}$ look like?

Earlier, we referred to subgroups as being "smaller." However, our definition does not imply that this has to be the case.

Theorem 4.8. Let G be a group of actions. Then $G \leq G$.

We refer to subgroups that are strictly smaller than the whole group as **proper subgroups**.

Lots of groups have been given formal names (e.g., D_4, R_4, etc.). However, not every group or subgroup has a name. In this case, it's useful to have notation to refer to specific subgroups.

Definition 4.9. Let G be a group of actions and let $g_1, \dots, g_n$ be distinct actions from G. We define $\langle g_1, \dots, g_n \rangle$ to be the smallest subgroup containing $g_1, \dots, g_n$. In this case, we call $\langle g_1, \dots, g_n \rangle$ the **subgroup generated by** $g_1, \dots, g_n$.

For example, consider $r, s, s' \in D_3$ (as defined in exercises 3.11 and 3.14). Then $\langle r, s \rangle = \langle s, s' \rangle = D_3$. Recall that R_4 is the subgroup of D_4 consisting of rotations of the square. Similarly, the group of rotations of an equilateral triangle is called R_3. Then using the r from D_3, we have $\langle r \rangle = R_3$, which is a subgroup of D_3.

Note that in Definition 4.9, we used a finite number of generators. There's no reason we have to do this. That is, we can consider groups/subgroups generated by infinitely many elements.

Exercise 4.10. Suppose $\{g_1, \dots, g_n\}$ is a generating set for a group G.

(a) Explain why $\{g_1^{-1}, \dots, g_n^{-1}\}$ is also a generating set for G.

(b) How does the Cayley diagram for G with generating set $\{g_1, \dots, g_n\}$ compare to the Cayley diagram with generating set $\{g_1^{-1}, \dots, g_n^{-1}\}$?

Exercise 4.11. Consider $\text{Spin}_{1\times 2}$.

(a) Can you find the Cayley diagram for $\langle t_1 \rangle$ as a subgroup of $\text{Spin}_{1\times 2}$?

(b) Write down all the actions of the subgroup $\langle t_1, t_2 \rangle$. Write them as words in t_1 and t_2. Can you find the Cayley diagram for $\langle t_1, t_2 \rangle$ as a subgroup of $\text{Spin}_{1\times 2}$? Can you find a clone for $\langle t_1, t_2 \rangle$?

One of the benefits of Cayley diagrams is that they are useful for visualizing subgroups. However, recall that if we change our set of generators, we might get a very different looking Cayley diagram. The upshot of this is that we may be able to see a subgroup in one Cayley diagram for a given group, but not be able to see it in a Cayley diagram with a different set of arrows.

Exercise 4.12. We currently have two different Cayley diagrams for D_3 (see Exercises 3.11 and 3.14).

(a) Can you find the Cayley diagram for $\langle e\rangle$ as a subgroup of D_3? Can you see it in both Cayley diagrams for D_3? Can you find all the clones?

(b) Can you find the Cayley diagram for $\langle r\rangle = R_3$ as a subgroup of D_3? Can you see it in both Cayley diagrams? Can you find all the clones?

(c) Find the Cayley diagrams for $\langle s\rangle$ and $\langle s'\rangle$ as subgroups of D_3. Can you see them in both Cayley diagrams for D_3? Can you find all the clones?

Exercise 4.13. Consider D_4. Let h be the action that reflects (i.e., flips over) the square over the horizontal midline and let v be the action that reflects the square over the vertical midline. Also, recall that r^2 is shorthand for the action rr that does r twice in a row. Which of the following are subgroups of D_4? In each case, justify your answer. If a subset is a subgroup, try to find a minimal set of generators. Also, determine whether you can see the subgroups in our Cayley diagram for D_4.

(a) $\{e, r^2\}$

(b) $\{e, h\}$

(c) $\{e, h, v\}$

(d) $\{e, h, v, r^2\}$

The subgroup in Exercise 4.13(d) is often referred to as the **Klein four-group** and is denoted by V_4.

Exercise 4.14. Draw the Cayley diagram for V_4 using $\{v, h\}$ as our set of generators.

Let's introduce a group we haven't seen yet. We define the **quaternion group** to be the group $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$ having the Cayley diagram with generators $i, j, -1$ given in Figure 4.1. In this case, 1 is the do-nothing action.

Notice that I didn't mention what the actions actually do. For now, let's not worry about that. The relationship between the arrows and vertices tells us everything we need to know. Also, let's take it for granted that Q_8 actually is a group.

Exercise 4.15. Consider the Cayley diagram for Q_8 given in Figure 4.1.

(a) Which arrows correspond to which generators in our Cayley diagram for Q_8?

(b) What is i^2 equal to? That is, what element of $\{1, -1, i, -i, j, -j, k, -k\}$ is i^2 equal to? How about i^3, i^4, and i^5?

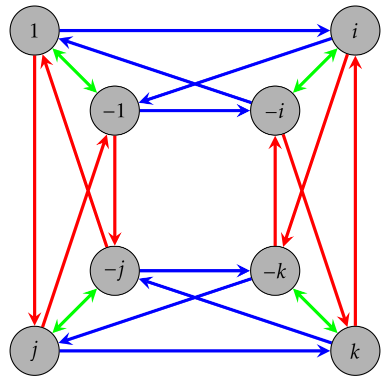

Figure 4.1. Cayley diagram for Q_8 with generating set $\{-1, i, j\}$.

Figure 4.2. Cayley diagram for S_2 with generator s.

(c) What are j^2, j^3, j^4, and j^5 equal to?

(d) What is $(-1)^2$ equal to?

(e) What is ij equal to? How about ji?

(f) Can you determine what k^2 and ik are equal to?

(g) Can you identify a generating set consisting of only two elements? Can you find more than one?

(h) What subgroups of Q_8 can you see in the Cayley diagram in Figure 4.1?

(i) Find a subgroup of Q_8 that you cannot see in the Cayley diagram.

4.2 Isomorphisms

By now you've probably seen enough examples of Cayley diagrams to witness some patterns appearing over and over again. One of the things you've probably noticed is that parts of some Cayley diagrams look just like parts of other Cayley diagrams.

Recall from Exercise 3.9 that S_2 is the group that acts on two coins by swapping their positions (but not flipping them over). We defined s to be the action that swaps the left and right coins and as usual e is the do-nothing action. The Cayley diagram for S_2 with generator s is given in Figure 4.2.

If you look back at all the Cayley diagrams you've encountered, you'll notice that many of them contained chunks that resemble the Cayley diagram for S_2 with generator s. In particular, in the Cayley diagrams for $\text{Spin}_{1\times 2}$, D_3, D_4, and Q_8 that we've seen, it is easy to identify the portions that "look like" S_2. For example, if you isolate the Cayley diagram for the subgroup $\langle -1\rangle = \{1, -1\}$ in Q_8, we see that it looks just like the

Cayley diagram for S_2, except the labels are not identical. The clones of the subgroup $\langle -1\rangle = \{1,-1\}$ in Q_8 look like S_2, as well, but they do not contain the do-nothing action.

The one thing that is different about the Cayley diagram for S_2 and the Cayley diagram for $\langle -1\rangle$ is that the labels are different. If we set the Cayley diagram for S_2 on top of the Cayley diagram for $\langle -1\rangle$ such that the do-nothing actions match up, then s and -1 would correspond to each other. In other words, the two Cayley diagrams are identical up to relabeling the vertices.

In this case, we say that S_2 and the subgroup $\langle -1\rangle$ of Q_8 are **isomorphic** under the correspondence $e \leftrightarrow 1$ and $s \leftrightarrow -1$. This one-to-one correspondence between the two groups is called an **isomorphism**, which is depicted in Figure 4.3. Note that I've recolored the arrow in S_2 so that it matches the corresponding arrow color of $\langle -1\rangle$. This isn't necessary, but it makes the correspondence more obvious.

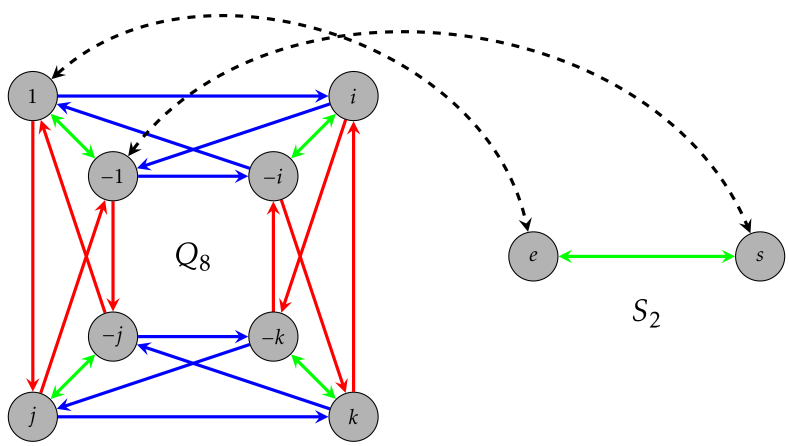

Figure 4.3. Isomorphism between $\langle -1\rangle \leq Q_8$ and S_2.

What this means is that these two groups have the same structure and characteristics. Or, in other words, these two groups essentially do the "same kind" of thing. Clearly, the two do-nothing actions behave the same way. Also, s and -1 both have the property that doing the action twice results in having done nothing (i.e., each element is its own reverse). Since there are only two elements, there isn't anything else to check. In groups with more elements, things can get much more complicated.

It is important to point out that S_2 and $\langle -1\rangle$ (in Q_8) are not equal. But they have the same structure. Identifying when two groups have the same structure (i.e., isomorphic) is an important pursuit in group theory.

If you look at the original Cayley diagram for $\mathrm{Spin}_{1\times 2}$ (with generators s, t_1, t_2), we can see three subgroups that look like S_2; namely $\langle s\rangle$, $\langle t_1\rangle$, and $\langle t_2\rangle$. Each of these three subgroups is isomorphic to S_2.

There is one serious potential for confusion here. Notice that there is an s in S_2 and an s in $\mathrm{Spin}_{1\times 2}$. Despite having identical names, they are not the same element. Since we only have 26 letters in our alphabet this sort of thing is unavoidable. Under the isomorphism between S_2 and the subgroup $\langle s\rangle$ in $\mathrm{Spin}_{1\times 2}$, the two elements with the same name match up. That is, these two elements are the ones in each group with the same behavior.

Exercise 4.16. Can you find any other subgroups or groups that are isomorphic to S_2?

Let's write down an official definition of isomorphic.

Definition 4.17. Let G and G' be two groups. We say that G and G' are **isomorphic** if there exist generating sets S and S' for G and G', respectively, such that the corresponding Cayley diagrams are identical where we ignore the labels on the vertices and recolor the edges if necessary. In this case, we write $G \cong G'$. Otherwise, we say that G and G' are not isomorphic. If G and G' are isomorphic, then the one-to-one correspondence determined by matching up the corresponding generators and respecting arrow paths is called an **isomorphism**.

The last sentence in the definition above might be a bit much to handle at the moment, but as we construct more examples, the concept should become clear. The general idea is to take two identical Cayley diagrams (ignoring labels) for G and G' and then set one on top of the other so that the vertices and arrows of the same color match up. This should be done so that the do-nothing actions correspond to each other. Then it becomes clear which actions in G correspond to which actions in G'. There might be many ways to do this.

Consider the group R_4 with generator r (rotation by 90° clockwise). Now, take a look at the Cayley diagram for Q_8 with generators $i, j, -1$. It should be easy to convince yourself that R_4 is isomorphic to both $\langle i \rangle = \{1, i, -i, -1\}$ and $\langle j \rangle = \{1, j, -j, -1\}$. However, you have to do some rearranging of one of the diagrams to set one on top of the other. Let's just focus on $\langle i \rangle$.

How do R_4 and $\langle i \rangle$ match up? We want to pair elements in each group with an element in the other group that has the same behavior. Clearly, e and 1 match up since these are the two do-nothing actions. Also, the reason why we noticed these two groups were isomorphic is because their Cayley diagrams looked the same. Since each Cayley diagram only had one arrow type determined by r and i, we should pair these two elements. Now, following the arrows around the diagram, we see that r^2 must pair with $i^2 = -1$ and r^3 corresponds to $i^3 = -i$. In summary, the isomorphism between R_4 and $\langle i \rangle$ (in Q_8) is given by $e \leftrightarrow 1$, $r \leftrightarrow i$, $r^2 \leftrightarrow -1$, and $r^3 \leftrightarrow -i$, which is depicted in Figure 4.4. Note that this time we have not recolored the edges so that they match. Nonetheless, the correspondence should be clear.

Now, take a look at the Cayley diagram for D_4 with generating set $\{r, s\}$. As we noticed in Exercise 4.2, R_4 is a subgroup of D_4. We could say that this subgroup is isomorphic to R_4, but in this case, we can say something even stronger: they are equal!

Before continuing, we need to emphasize an important point. If the Cayley diagram for one group does not look like the Cayley diagram for another group, then that does *not* immediately imply that the groups are not isomorphic. The issue is that perhaps we could choose appropriate generating sets for each group so that the Cayley diagrams do look alike. For example, notice that our standard Cayley diagram for R_4 does not look like the Cayley diagram that you constructed for V_4 in Exercise 4.14. This does *not* imply that these two groups are not isomorphic. We would need to do some more work in order to determine whether the two groups are isomorphic or not. You will get a chance to do this in Exercise 4.21.

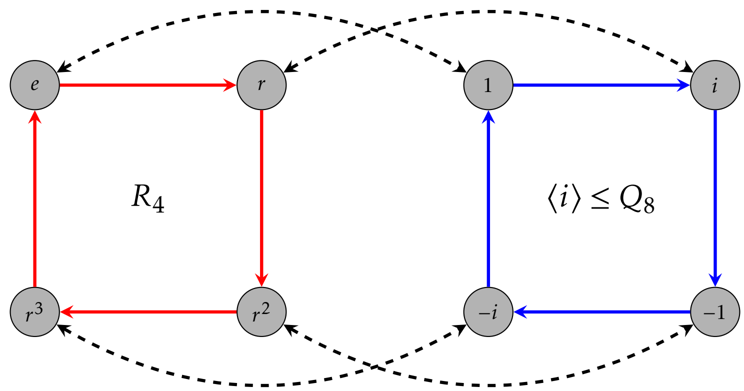

Figure 4.4. Isomorphism between $\langle i \rangle \leq Q_8$ and R_4.

It turns out that there is a fancy word for the size of a group.

Definition 4.18. If G is a group with n distinct actions, then we say that G has **order** n and write $|G| = n$. If G contains infinitely many elements, then we say G has infinite order and write $|G| = \infty$.

Exercise 4.19. Find the orders of the following groups: S_2, $\text{Spin}_{1\times 2}$, $\text{Spin}_{3\times 3}$, R_4, D_3, D_4, V_4, and Q_8.

Theorem 4.20. Suppose G and G' are two groups of actions such that $G \cong G'$. Then $|G| = |G'|$.

Unfortunately, the converse of the previous theorem is not true in general. That is, two groups that have the same order may or may not be isomorphic.

Loosely speaking, if one group has a property that the other does not have, then the two groups cannot be isomorphic. For example, if one group has the property that every pair of actions commutes (i.e., the order† of the actions does not matter), but another group has a pair of actions that do not commute, then the two groups cannot be isomorphic. Moreover, if one group contains an action that requires a minimum of k applications to get back to the do-nothing action, but a second group does not have such an element, then the two groups cannot be isomorphic.

Justifying these two claims takes a bit of work and for now, we'll put that on hold. For the time being, if you don't see why these claims about when two groups are not isomorphic are true, just take them on faith and we will return to the issue in a later chapter. Feel free to use these ideas in the exercises that follow.

Problem 4.21. Determine whether R_4 and V_4 are isomorphic. Justify your answer. If they are isomorphic, specify the isomorphism by listing the correspondence of elements. If they are not isomorphic, explain why.

Problem 4.22. Consider the group given by the Cayley diagram for R_6 that was given in Exercise 3.15. We can think of R_6 as the rotation group for a regular hexagon. Determine

†Don't confuse the word "order" in this sentence with the order of a group.

whether R_6 and D_3 are isomorphic. Justify your answer. If they are isomorphic, specify the isomorphism by listing the correspondence of elements. If they are not isomorphic, explain why.

Exercise 4.23. Consider two light switches on a wall side by side. Consider the group of actions that consists of all possible actions that you can do to the two light switches. For example, one action is toggle the left light switch while leaving the right alone. Let's call this group L_2.

(a) How many distinct actions does L_2 have?

(b) Can you find a minimal generating set for L_2? If so, give these actions names and then write all of the actions of L_2 as words in your generator(s).

(c) Using your generators from part (b), draw a Cayley diagram for L_2.

Problem 4.24. Determine whether L_2 and V_4 are isomorphic. Justify your answer. If they are isomorphic, specify the isomorphism by listing the correspondence of elements. If they are not isomorphic, explain why.

Problem 4.25. Determine whether Q_8 and D_4 are isomorphic. Justify your answer. If they are isomorphic, specify the isomorphism by listing the correspondence of elements. If they are not isomorphic, explain why.

Problem 4.26. Determine whether $\text{Spin}_{1\times 2}$ and D_4 are isomorphic. Justify your answer. If they are isomorphic, specify the isomorphism by listing the correspondence of elements. If they are not isomorphic, explain why.

Exercise 4.27. Consider the group that acts on three coins that are in a row by rearranging their positions (but not flipping them over). This group is called S_3. Number the positions of the coins (not the coins themselves) 1, 2, 3 from left to right. Let s_1 be the action that swaps the coins in positions 1 and 2 and let s_2 be the action that swaps the coins in positions 2 and 3.

(a) The group S_3 consists of 6 actions, which we can generate with s_1 and s_2. Write all 6 actions as words in s_1 and s_2.

(b) Using s_1 and s_2 as generators, draw a Cayley diagram for S_3.

Problem 4.28. Determine whether S_3 and D_3 are isomorphic. Justify your answer. If they are isomorphic, specify the isomorphism by listing the correspondence of elements. If they are not isomorphic, explain why. Don't forget that we've drawn two different Cayley diagrams for D_3.