Chapter 6

Products and Quotients of Groups

6.1 Products of Groups

In this section, we will discuss a method for using existing groups as building blocks to form new groups.

Suppose $(G,*)$ and $(H,\odot)$ are two groups. Recall that the Cartesian product of G and H is defined to be

$$G \times H = \{(g,h) \mid g \in G, h \in H\}$$

Using the binary operations for the groups G and H, we can define a binary operation on the set $G \times H$. Define $\star$ on $G \times H$ via

$$(g_1,h_1) \star (g_2,h_2) = (g_1 * g_2, h_1 \odot h_2).$$

This looks fancier than it is. We're just doing the operation of each group in the appropriate component. It turns out that $(G \times H, \star)$ is a group.

Theorem 6.1. Suppose $(G,*)$ and $(H,\odot)$ are two groups, where e and e' are the identity elements of G and H, respectively. Then $(G \times H, \star)$ is a group, where $\star$ is defined as above. Moreover, (e,e') is the identity of $G \times H$ and the inverse of $(g,h) \in G \times H$ is given by $(g,h)^{-1} = (g^{-1}, h^{-1})$.

We refer to $G \times H$ as the **direct product** of the groups G and H. In this case, each of G and H is called a **factor** of the direct product. We often abbreviate $(g_1,h_1) \star (g_2,h_2) = (g_1 * g_2, h_1 \odot h_2)$ by $(g_1,h_1)(g_2,h_2) = (g_1g_2, h_1h_2)$. One exception to this is if we are using the operation of addition in each component. For example, consider $\mathbb{Z}_4 \times \mathbb{Z}_2$ under the operation of addition mod 4 in the first component and addition mod 2 in the second component. Then

$$\mathbb{Z}_4 \times \mathbb{Z}_2 = \{(0,0),(1,0),(2,0),(3,0),(0,1),(1,1),(2,1),(3,1)\}.$$

In this case, we will use additive notation in $\mathbb{Z}_4 \times \mathbb{Z}_2$. For example, in $\mathbb{Z}_4 \times \mathbb{Z}_2$ we have

$$(2,1) + (3,1) = (1,0)$$

and

$$(1,0)+(2,1)=(3,1).$$

Moreover, the identity of the group is $(0,0)$. As an example, the inverse of $(1,1)$ is $(3,1)$ since $(1,1)+(3,1)=(0,0)$. There is a very natural generating set for $\mathbb{Z}_4\times\mathbb{Z}_2$, namely, $\{(1,0),(0,1)\}$ since $1\in\mathbb{Z}_4$ and $1\in\mathbb{Z}_2$ generate $\mathbb{Z}_4$ and $\mathbb{Z}_2$, respectively. The corresponding Cayley diagram is given in Figure 6.1.

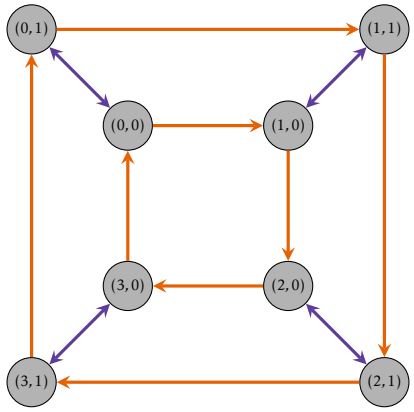


Figure 6.1. Cayley diagram for $\mathbb{Z}_4\times\mathbb{Z}_2$ with generating set $\{(1,0),(0,1)\}$.

Problem 6.2. Consider the group $\mathbb{Z}_4\times\mathbb{Z}_2$. Is this group abelian? Is the group cyclic? Determine whether $\mathbb{Z}_4\times\mathbb{Z}_2$ is isomorphic to any of D_4, Q_8, $\mathbb{Z}_8$, or L_3.

The upshot of the previous problem is that there are at least five groups of order 8 up to isomorphism. It turns out that there are exactly five groups of order 8 up to isomorphism. In particular, every group of order 8 is isomorphic to one of the following groups: $\mathbb{Z}_8$, $\mathbb{Z}_4\times\mathbb{Z}_2$, L_3, D_4, and Q_8. Note that $R_8\cong\mathbb{Z}_8$ and $\mathrm{Spin}_{1\times 2}\cong D_4$. Three of the isomorphism classes correspond to abelian groups while the other two correspond to non-abelian groups. Unfortunately, we will not develop the tools necessary to prove that this classification is complete.

The next two theorems should not be terribly surprising.

Theorem 6.3. If G_1 and G_2 are groups, then $G_1\times G_2\cong G_2\times G_1$.

Theorem 6.4. Suppose G_1 and G_2 are groups with identities e_1 and e_2, respectively. Then $\{e_1\}\times G_2\cong G_2$ and $G_1\times\{e_2\}\cong G_1$.

There's no reason we can't take the direct product of more than two groups. If $A_1,A_2,\ldots,A_n$ is a collection of sets, we define

$$\prod_{i=1}^{n}A_i:=A_1\times A_2\times\cdots\times A_n.$$

Each element of $\prod_{i=1}^{n}A_i$ is of the form $(a_1,a_2,\ldots,a_n)$, where $a_i\in A_i$.

Theorem 6.5. Let $G_1, G_2, \ldots, G_n$ be groups. For $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in \prod_{i=1}^{n} G_i$, define

$$(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n) = (a_1 b_1, a_2 b_2, \ldots, a_n b_n).$$

Then $\prod_{i=1}^{n} G_i$, the **direct product** of $G_1, \ldots, G_n$, is a group under this binary operation.

One way to think about direct products is that we can navigate the product by navigating each factor simultaneously but independently. Computing the order of a group that is a direct product is straightforward.

Theorem 6.6. Let $G_1, G_2, \ldots, G_n$ be finite groups. Then

$$|G_1 \times G_2 \times \cdots \times G_n| = |G_1| \cdot |G_2| \cdots |G_n|.$$

Theorem 6.7. Let $G_1, G_2, \ldots, G_n$ be groups. Then $|G_1 \times G_2 \times \cdots \times G_n|$ is infinite if and only if at least one $|G_i|$ is infinite.

The following theorem should be clear.

Theorem 6.8. Let $G_1, G_2, \ldots, G_n$ be groups. Then $\prod_{i=1}^{n} G_i$ is abelian if and only if each G_i is abelian.

Let's play with a few more examples.

Problem 6.9. Draw the Cayley diagram for $\mathbb{Z}_2 \times \mathbb{Z}_3$ using $\{(1,0), (0,1)\}$ as the generating set. Is $\mathbb{Z}_2 \times \mathbb{Z}_3$ an abelian group? Is it cyclic? What familiar group is $\mathbb{Z}_2 \times \mathbb{Z}_3$ isomorphic to?

Problem 6.10. Consider $\mathbb{Z}_2 \times \mathbb{Z}_2$ under the operation of addition mod 2 in each component. Find a generating set for $\mathbb{Z}_2 \times \mathbb{Z}_2$ and then create a Cayley diagram for this group. What well-known group is $\mathbb{Z}_2 \times \mathbb{Z}_2$ isomorphic to?

Consider the similarities and differences between $\mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. Both groups are abelian by Theorem 6.8, but only the former is cyclic. Here's another exercise.

Problem 6.11. Consider the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Find a generating set for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and then create a Cayley diagram for this group. Is there a group that we have seen before that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ isomorphic to?

The next theorem tells us how to compute the order of an element in a direct product of groups.

Theorem 6.12. Suppose $G_1, G_2, \ldots, G_n$ are groups and let $(g_1, g_2, \ldots, g_n) \in \prod_{i=1}^{n} G_i$. If $|g_i| = r_i < \infty$, then $|(g_1, g_2, \ldots, g_n)| = \operatorname{lcm}(r_1, r_2, \ldots, r_n)$.

Problem 6.13. Find the order of each of the following elements.

(a) $(6, 5) \in \mathbb{Z}_{12} \times \mathbb{Z}_7$.

(b) $(r, i) \in D_3 \times Q_8$.

(c) $((1,2)(3,4),3) \in S_4 \times \mathbb{Z}_{15}$.

Problem 6.14. Find the largest possible order of elements in each of the following groups.

(a) $\mathbb{Z}_6 \times \mathbb{Z}_8$

(b) $\mathbb{Z}_9 \times \mathbb{Z}_{12}$

(c) $\mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{15}$

Theorem 6.15. The group $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic if and only if m and n are relatively prime.

Corollary 6.16. The group $\mathbb{Z}_m \times \mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime.

The previous results can be extended to more than two factors.

Theorem 6.17. The group $\prod_{i=1}^{n} \mathbb{Z}_{m_i}$ is cyclic and isomorphic to $\mathbb{Z}_{m_1 m_2 \cdots m_n}$ if and only if every pair from the collection $\{m_1, m_2, \ldots, m_n\}$ is relatively prime.

Problem 6.18. Determine whether each of the following groups is cyclic.

(a) $\mathbb{Z}_7 \times \mathbb{Z}_8$

(b) $\mathbb{Z}_7 \times \mathbb{Z}_7$

(c) $\mathbb{Z}_2 \times \mathbb{Z}_7 \times \mathbb{Z}_8$

(d) $\mathbb{Z}_5 \times \mathbb{Z}_7 \times \mathbb{Z}_8$

Theorem 6.19. Suppose $n = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$, where each p_i is a distinct prime number. Then

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \cdots \times \mathbb{Z}_{p_r^{n_r}}.$$

The next theorem tells us that the direct product of subgroups is always a subgroup.

Theorem 6.20. Suppose G_1 and G_2 are groups such that $H_1 \leq G_1$ and $H_2 \leq G_2$. Then $H_1 \times H_2 \leq G_1 \times G_2$.

However, not every subgroup of a direct product has the form above.

Problem 6.21. Find an example that illustrates that not every subgroup of a direct product is the direct product of subgroups of the factors.

Problem 6.22. Can we extend Theorem 6.20 to normal subgroups? That is, if $H_1 \trianglelefteq G_1$ and $H_2 \trianglelefteq G_2$, is it the case that $H_1 \times H_2 \trianglelefteq G_1 \times G_2$? If so, prove it. Otherwise, provide a counterexample.

The next theorem describes precisely the structure of finite abelian groups. We will omit its proof, but allow ourselves to utilize it as needed.

Theorem 6.23 (Fundamental Theorem of Finitely Generated Abelian Groups)**.** Every finitely generated abelian group G is isomorphic to a direct product of cyclic groups of the form

$$\mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \cdots \times \mathbb{Z}_{p_r^{n_r}} \times \mathbb{Z}^k,$$

where each p_i is a prime number (not necessarily distinct). The product is unique up to rearrangement of the factors.

Note that the number k is called the **Betti number**. A finitely generated abelian group is finite if and only if the Betti number is 0.

Problem 6.24. Find all abelian groups up to isomorphism of order 8. How many different groups up to isomorphism (both abelian and non-abelian) have we seen and what are they?

Problem 6.25. Find all abelian groups up to isomorphism for each of the following orders.

(a) 16

(b) 12

(c) 25

(d) 30

(e) 60

6.2 Quotients of Groups

In the previous section, we discussed a method for constructing "larger" groups from "smaller" groups using a direct product construction. In this section, we will in some sense do the opposite.

Problem 5.28 hinted that if $H \leq G$ and we arrange the group table according to the left cosets of H, then the group table will have checkerboard pattern if and only if H is normal in G (i.e., the left and right cosets of H are the same). For example, see the colored table prior to Problem 5.3 versus the ones you created in Exercises 5.3, 5.4. If we have the checkerboard pattern in the group table that arises from a normal subgroup, then by "gluing together" the colored blocks, we obtain a group table for a smaller group that has the cosets as the elements.

For example, let's consider $H = \langle -1 \rangle \leq Q_8$. Problem 5.4 showed us that K is normal Q_8. The left (and right) cosets of H in Q_8 are

$$K = \{1, -1\}, iH = \{i, -i\}, jH = \{j, -j\}, \text{ and } kH = \{k, -k\}.$$

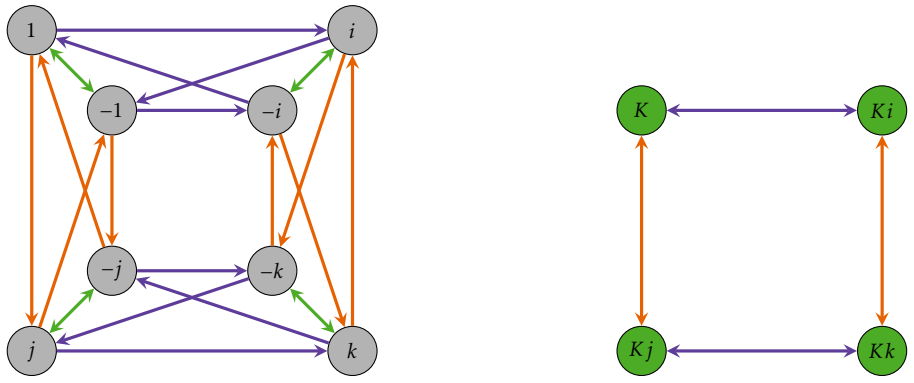
As you found in Problem 5.4, if we arrange the rows and columns of Q_8 according to these cosets, we obtain the following group table.

$*$	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

If we consider the 2×2 blocks as elements, it appears that we have a group table for a group with 4 elements. Closer inspection reveals that this looks like the table for V_4 (compare with Figure 3.8). If the table of 2×2 blocks is going to represent a group, we need to understand the binary operation. How do we "multiply" cosets? For example, the table suggests that the coset jH (colored in purple) times the coset iH (colored in orange) is equal to kH (colored in light purple) despite the fact that $ji = -k \neq k$. Yet, it is true that the product $ji = -k$ is an element in the coset kH. In fact, if we look closely at the table, we see that if we pick any two cosets, the product of any element of the first coset times any element of the second coset will always result in an element in the same coset regardless of which representatives we chose.

In other words, it looks like we can multiply cosets by choosing any representative from each coset and then seeing what coset the product of the representatives lies in. However, it is important to point out that this will only work if we have a checkerboard pattern of cosets, which we have seen evidence of only happening when the corresponding subgroup is normal.

Before continuing, let's continue tinkering with the same example. Consider the Cayley diagram for Q_8 with generators $\{i, j, -1\}$ that is given in Figure 6.2(a).



(a) (b)

Figure 6.2. The left subfigure shows the Cayley diagram for Q_8 with generating set $\{i, j, -1\}$. The right subfigure shows the collapsed Cayley diagram for Q_8 according to the right cosets of $K = \langle -1 \rangle$.

We can visualize the right cosets of K as the four clumps of vertices connected together with the two-way green arrows. These are the clones of K. In this case, we are also seeing

the left cosets since K is normal in Q_8. If we collapse the right cosets onto each other and collapse the corresponding arrows, we obtain the diagram given in Figure 6.2(b). It is clear that this diagram is the Cayley diagram for a group that is isomorphic to V_4.

This process of collapsing a Cayley diagram for a group G according to the right cosets of a subgroup H is called the **quotient process**. The vertices in the resulting diagram are the right cosets (i.e., clones) of H. If there is an arrow from g_1 to g_2 in the Cayley diagram for G such that g_1 and g_2 are in different right cosets, then there is an arrow of the same color from the right coset containing g_1 (i.e., Hg_1) to the right coset containing g_2 (i.e., Hg_2).

For reasons we will understand shortly, applying the quotient process to G according to a subgroup H results in the Cayley diagram for a group if and only if H is normal in G.

Problem 6.26. Let's see what happens if we attempt the quotient process for a subgroup that is not normal. Consider $H = \langle s \rangle \leq D_3$. In Problem 5.2, we discovered that the left cosets of H are not the same as the right cosets of H. This implies that H is not normal in D_3. Consider the standard Cayley diagram for D_3 that uses the generators r and s. Draw the diagram that results from attempting the quotient process on D_3 using the subgroup H. Explain why this diagram cannot be the diagram for a group.

The problem that arises in Problem 6.26 is that if the same arrow types (i.e., those representing the same generator) leaving a right coset do not point at elements in the same right coset, attempting the quotient process will result in a diagram that cannot be a Cayley diagram for a group since we have more than one arrow of the same type leaving a vertex. In Figure 6.3(a), we illustrate what goes wrong if all the arrows for a generator pointing out of a right coset (i.e., clone) do not unanimously point to elements in the same right coset. In Figure 6.3(b), all the arrows point to elements in the same right coset, and in this case, it appears that everything works out just fine.

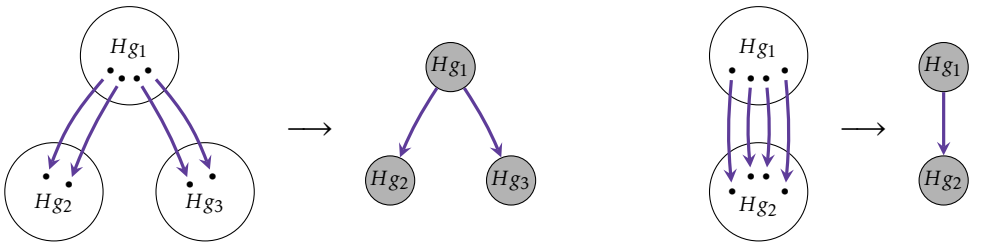


Figure 6.3. In the left subfigure, blue arrows go from elements of the right coset Hg_1 to elements of *multiple* left cosets, which results in ambiguous blue arrows in the diagram resulting from the quotient process. This implies that right coset multiplication is not well defined in this case. In the right subfigure, blue arrows go from elements of the right coset Hg_1 to elements inside a *unique* right coset, which does not result in any ambiguity in the diagram resulting from the quotient process.

Problem 6.27. In Problem 5.3, we learned that the subgroup $K = \langle r \rangle$ is normal in D_3 since the left cosets are equal to the right cosets. Note that this follows immediately from

Theorem 5.33 since $[D_3 : K] = 2$. Draw the diagram that results from performing the quotient process to D_3 using the subgroup K. Does the resulting diagram represent a group? If so, what group is it isomorphic to?

Problem 6.28. Consider the left Cayley diagram for A_4 given in Problem 4.122.

(a) Let H be the subgroup generated by the element that corresponds to the orange arrow. Draw the diagram that results from performing the quotient process to A_4 using the subgroup H. Does the resulting diagram represent a group? If so, what group is it isomorphic to? Compare with Problem 5.32.

(b) Let K be the subgroup generated by the element that corresponds to the purple arrow. Draw the diagram that results from performing the quotient process to A_4 using the subgroup K. Does the resulting diagram represent a group? If so, what group is it isomorphic to?

Problem 6.29. Consider the Cayley diagram for $\mathbb{Z}_4 \times \mathbb{Z}_2$ given in Figure 6.1.

(a) Let $H = \langle (1,0) \rangle$. Draw the diagram that results from performing the quotient process to $\mathbb{Z}_4 \times \mathbb{Z}_2$ using the subgroup H. Does the resulting diagram represent a group? If so, what group is it isomorphic to?

(b) Let $K = \langle (0,1) \rangle$. Draw the diagram that results from performing the quotient process to $\mathbb{Z}_4 \times \mathbb{Z}_2$ using the subgroup K. Does the resulting diagram represent a group? If so, what group is it isomorphic to?

Now, suppose G is an arbitrary group and let $H \leq G$. Consider the set of left cosets of H. We would like to define

$$(aH)(bH) = (ab)H.$$

The natural question to ask is whether this operation is well defined. That is, does the result of multiplying two left cosets depend on our choice of representatives? More specifically, suppose $c \in aH$ and $d \in bH$. Then $cH = aH$ and $dH = bH$. According to the operation defined above, $(cH)(dH) = cdH$. It better be the case that $cdH = abH$, otherwise the operation is not well defined.

Problem 6.30. Let $H = \langle s \rangle \leq D_3$. Find specific examples of $a, b, c, d \in D_3$ such that

$$(aH)(bH) \neq (cH)(dH)$$

even though $aH = cH$ and $bH = dH$.

Theorem 6.31. Let G be a group and let $H \leq G$. Then left coset multiplication (as defined above) is well defined if and only if $H \trianglelefteq G$.

Theorem 6.32. Let G be a group and let $H \trianglelefteq G$. Then the set of left cosets of H in G forms a group under left coset multiplication.

The group from Theorem 6.32 is denoted by G/H, read "G mod H", and is referred to as the **quotient group** (or **factor group**) of G by H. If G is a finite group, then G/H is exactly the group that arises from "gluing together" the colored blocks in a checkerboard-patterned group table. It's also the group that we get after applying the quotient process to the Cayley diagram. It's important to point out once more that this only works properly if H is a normal subgroup.

Recall that Theorem 5.30 tells us that if G is abelian, then every subgroup is normal. This implies that when G is abelian, G/H is a well-defined group for every subgroup H of G. However, it is not necessary for G to be abelian in order for G/H to be a well-defined group. The quotient group $Q_8/\langle -1\rangle$ is an example where this happens.

The next theorem tells us how to compute the order of a quotient group.

Theorem 6.33. Let G be a group and let $H \trianglelefteq G$. Then $|G/H| = [G : H]$. In particular, if G is finite, then $|G/H| = |G|/|H|$.

It's important to point out that the order of a quotient group might be finite even if G has infinite order.

Problem 6.34. Consider the group $(\mathbb{Z}, +)$. Since $\mathbb{Z}$ is abelian, every subgroup is normal. For example, $4\mathbb{Z} \trianglelefteq \mathbb{Z}$, which implies that $\mathbb{Z}/4\mathbb{Z}$ is a well-defined quotient group. Moreover, both $\mathbb{Z}$ and $4\mathbb{Z}$ have infinite order. What is $|\mathbb{Z}/4\mathbb{Z}|$ equal to? Can you determine what well-known group $\mathbb{Z}/4\mathbb{Z}$ is isomorphic to?

Suppose G is a group and $H \trianglelefteq G$, so that G/H is a group. Recall that the elements of the group G/H are the left cosets of H, which are of the form aH where $a \in G$. The operation of the group is defined via

$$(aH)(bH) = abH.$$

Moreover, the identity in G/H is $eH = H$ since $(aH)(eH) = aH$. By Corollary 4.20 $|aH| = k$ if and only if $(aH)^k = H$ and k is the smallest such positive exponent with this property. But notice that $(aH)^k = a^k H$. So, in order to compute the order of aH, we need to find the smallest positive exponent k such that $a^k H = H$, but $a^k H = H$ exactly when a^k is in H. The upshot is that to find the order of aH in G/H, we need the smallest positive k such that a^k is in H.

Problem 6.35. Find the order of the given element in the quotient group. You may assume that we are taking the quotient by a normal subgroup.

(a) $s\langle r\rangle \in D_4/\langle r\rangle$

(b) $j\langle -1\rangle \in Q_8/\langle -1\rangle$

(c) $5 + \langle 4\rangle \in \mathbb{Z}_{12}/\langle 4\rangle$

(d) $(2,1) + \langle (1,1)\rangle \in (\mathbb{Z}_3 \times \mathbb{Z}_6)/\langle (1,1)\rangle$

(e) $(1,3) + \langle (0,2)\rangle \in (\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle (0,2)\rangle$

Problem 6.36. For each quotient group below, describe the group. If possible, state what group each is isomorphic to. You may assume that we are taking the quotient by a normal subgroup.

(a) $Q_8/\langle -1\rangle$

(b) $Q_8/\langle i\rangle$

(c) $\mathbb{Z}_4/\langle 2\rangle$

(d) $V_4/\langle h\rangle$

(e) $A_4/\langle (1,2)(3,4),(1,3)(2,4)\rangle$

(f) $(\mathbb{Z}_2\times\mathbb{Z}_2)/\langle (1,1)\rangle$

(g) $\mathbb{Z}/4\mathbb{Z}$

(h) S_4/A_4

(i) $(\mathbb{Z}_4\times\mathbb{Z}_2)/(\{0\}\times\mathbb{Z}_2)$

Problem 6.37. Compute the order of every element in the quotient group $(\mathbb{Z}_2\times\mathbb{Z}_4)/\langle (0,2)\rangle$. What well-known group is $(\mathbb{Z}_2\times\mathbb{Z}_4)/\langle (0,2)\rangle$ isomorphic to?

Theorem 6.38. Let G be a group. Then

(a) $G/\{e\}\cong G$

(b) $G/G\cong\{e\}$

Theorem 6.39. We have the following.

(a) For $n\geq 2$, $S_n/A_n\cong\mathbb{Z}_2$.

(b) For all $n\in\mathbb{N}$, $\mathbb{Z}/n\mathbb{Z}\cong\mathbb{Z}_n$.

(c) For all $n\in\mathbb{N}$, $\mathbb{R}/n\mathbb{R}\cong\{e\}$.

Theorem 6.40. Let G be a group and let $H\trianglelefteq G$. If G is abelian, then so is G/H.

Problem 6.41. Show that the converse of the previous theorem is not true by providing a specific counterexample.

Problem 6.42. Consider the quotient group $(\mathbb{Z}_4\times\mathbb{Z}_6)/\langle (0,1)\rangle$.

(a) What is the order of $(\mathbb{Z}_4\times\mathbb{Z}_6)/\langle (0,1)\rangle$?

(b) Is the group abelian? Why?

(c) Write down all the elements of $(\mathbb{Z}_4\times\mathbb{Z}_6)/\langle (0,1)\rangle$.

(d) Does one of the elements generate the group?

(e) What well-known group is $(\mathbb{Z}_4 \times \mathbb{Z}_6)/\langle(0,1)\rangle$ isomorphic to?

Theorem 6.43. Let G be a group and let $H \trianglelefteq G$. If G is cyclic, then so is G/H.

Problem 6.44. Show that the converse of the previous theorem is not true by providing a specific counterexample.

The next result should be straightforward to proof.

Theorem 6.45. If G is a group, then $Z(G) \trianglelefteq G$.

The next theorem is surprisingly useful at times.

Theorem 6.46. Let G be a group. If $G/Z(G)$ is cyclic, then G is abelian.

Problem 6.47. Is the converse of the previous theorem true? If so, prove it. Otherwise, provide a counterexample.

Theorem 6.48. If G is a group such that $|G| = pq$, where p and q are primes (not necessarily distinct), then either $Z(G) = \{e\}$ or G is abelian.

Problem 6.49. Recall that $\mathbb{Q}$ is a group under addition. Define $\mathbb{Q}^* := \mathbb{Q} \setminus \{0\}$. It is easy to verify that $\mathbb{Q}^*$ is a group under multiplication. Both groups are abelian, which implies that all subgroups are normal.

(a) Describe the quotient group $\mathbb{Q}/\langle -1\rangle$. In particular, what do the elements (cosets) look like?

(b) Describe the quotient group $\mathbb{Q}^*/\langle -1\rangle$. In particular, what do the elements (cosets) look like?

Here are few additional exercises. These ones are a bit tougher.

Problem 6.50. For each quotient group below, describe the group. If possible, state what group each is isomorphic to. You may assume that we are taking the quotient by a normal subgroup.

(a) $(\mathbb{Z}_4 \times \mathbb{Z}_6)/\langle(0,2)\rangle$

(b) $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(0,2)$

(c) $(\mathbb{Z} \times \mathbb{Z})/\langle(1,1)\rangle$

(d) $\mathbb{Q}/\langle 1\rangle$