Chapter 6

Limits

We are now prepared to dig into limits, which you are likely familiar with from calculus. However, chances are that you were never introduced to the formal definition.

Definition 6.1. Let $f : A \to \mathbb{R}$ be a function, where $A \subseteq \mathbb{R}$. The **limit** of f as x approaches a is L if the following two conditions hold:

1. The point a is an accumulation point of A, and
2. For every $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Notationally, we write this as

$$\boxed{\lim_{x \to a} f(x) = L.}$$

It turns out that limits are unique if they exist. You may assume this going forward.

Problem 6.2. Why do we require $0 < |x - a|$ in Definition 6.1?

Problem 6.3. Why do you think we require a to be an accumulation point of the domain of f? What happens if $a \in A$ but a is not an accumulation point of A (such points are called **isolated points** of A)?

Example 6.4. It should come as no surprise to you that $\lim_{x \to 5}(3x + 2) = 17$. Let's prove this using Definition 6.1. First, notice that the default domain of $f(x) = 3x + 2$ is the set of real numbers. So, any x-value we choose will be in the domain of the function. Now, let $\epsilon > 0$. Choose $\delta = \epsilon/3$. You'll see in a moment why this is a good choice for δ. Suppose $x \in \mathbb{R}$ such that $0 < |x - 5| < \delta$. We see that

$$|(3x + 2) - 17| = |3x - 15| = 3 \cdot |x - 5| < 3 \cdot \delta = 3 \cdot \epsilon/3 = \epsilon.$$

This proves the desired result.

Example 6.5. Let's try something a little more difficult. Let's prove that $\lim_{x \to 3} x^2 = 9$. As in the previous example, the default domain of our function is the set of real numbers. Our goal is to prove that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in \mathbb{R}$ such that

$0 < |x-3| < \delta$, then $|x^2 - 9| < \epsilon$. Let $\epsilon > 0$. We need to figure out what δ needs to be. Notice that

$$|x^2 - 9| = |x+3| \cdot |x-3|.$$

The quantity $|x-3|$ is something we can control with δ, but the quantity $|x+3|$ seems to be problematic.

To get a handle on what's going on, let's temporarily assume that $\delta = 1$ and suppose that $0 < |x-3| < 1$. This means that x is within 1 unit of 3. In other words, $2 < x < 4$. But this implies that $5 < x+3 < 7$, which in turn implies that $|x+3|$ is bounded above by 7. That is, $|x+3| < 7$ when $0 < |x-3| < 1$. It's easy to see that we still have $|x+3| < 7$ even if we choose δ smaller than 1. That is, we have $|x+3| < 7$ when $0 < |x-3| < \delta \leq 1$. Putting this altogether, if we suppose that $0 < |x-3| < \delta \leq 1$, then we can conclude that

$$|x^2 - 9| = |x+3| \cdot |x-3| < 7 \cdot |x-3|.$$

This work informs our choice of δ, but remember our scratch work above hinged on knowing that $\delta \leq 1$. If $\epsilon/7 \leq 1$, we should choose $\delta = \epsilon/7$. However, if $\epsilon/7 > 1$, the easiest thing to do is to just let $\delta = 1$. Let's button it all up.

Let $\epsilon > 0$. Choose $\delta = \min\{1, \epsilon/7\}$ and suppose $0 < |x-3| < \delta$. We see that

$$|x^2 - 9| = |x+3| \cdot |x-3| < 7 \cdot |x-3| < 7 \cdot \delta \leq \epsilon$$

since

$$7 \cdot \delta = \begin{cases} 7, & \text{if } \epsilon > 7 \\ 7 \cdot \epsilon/7, & \text{if } \epsilon \leq 7. \end{cases}$$

Therefore, $\lim_{x \to 3} x^2 = 9$, as expected.

Problem 6.6. Prove that $\lim\limits_{x \to 1}(17x - 42) = -25$ using Definition 6.1.

Problem 6.7. Prove that $\lim\limits_{x \to 2} x^3 = 8$ using Definition 6.1.

Problem 6.8. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} x, & \text{if } x \neq 0 \\ 17, & \text{if } x = 0. \end{cases}$$

Using Definition 6.1, prove that $\lim_{x \to 0} f(x) = 0$.

Problem 6.9. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \leq 0 \\ -1, & \text{if } x > 0. \end{cases}$$

Using Definition 6.1, prove that $\lim\limits_{x \to 0} f(x)$ does not exist.

Problem 6.10. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Using Definition 6.1, prove that $\lim_{x\to a} f(x)$ does not exist for all $a \in \mathbb{R}$.

Problem 6.11. Let $f : A \to \mathbb{R}$ be a function. Prove that if $\lim_{x\to a} f(x)$ exists, then the limit is unique.

The $\epsilon - \delta$ approach to a function f being continuous at $x = a$ (see Problem 5.5) looks awfully similar to the definition of the limit of f as x approaches a. Let's explore this a bit.

Problem 6.12. Explain the similarities and differences between the definitions of continuity at $x = a$ versus the limit as x approaches a. State a theorem about continuity involving limits. You will have to make a special statement about isolated points of the domain.

Perhaps not surprisingly, there is a nice connection between limits and sequences.

Problem 6.13. Let $f : A \to \mathbb{R}$ be a function and let a be an accumulation point of A. Then $\lim_{x\to a} f(x)$ exists if and only if for every sequence (x_n) in $A \setminus \{a\}$ converging to a, the sequence $(f(x_n))$ converges, in which case, $\lim_{x\to a} f(x)$ equals the limit of the sequence $(f(x_n))$. This is often written as

$$\lim_{x\to a} f(x) = \lim_{n\to\infty} f(x_n).$$

In order for limits to be a useful tool, we need to prove a few important facts.

Problem 6.14 (Limit Laws)**.** Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions. Prove each of the following using Definition 6.1.

(a) If $c \in \mathbb{R}$, then $\lim_{x\to a} c = c$.

(b) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then

$$\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x).$$

(c) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then

$$\lim_{x\to a}(f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x).$$

(d) If $c \in \mathbb{R}$ and $\lim_{x\to a} f(x)$ exists, then

$$\lim_{x\to a}(c \cdot f(x)) = c \cdot \lim_{x\to a} f(x).$$

(e) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and $\lim_{x\to a} g(x) \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

(f) If f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b).$$

The next problem is extremely useful. It allows us to simplify our calculations when computing limits.

Problem 6.15. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions and let a be an accumulation point of A. If there exists an open interval S such that $f(x) = g(x)$ for all $x \in (S \cap A) \setminus \{a\}$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

provided one of the limits exists.