Chapter 2

Preliminaries

In this chapter, we summarize some background material we need to be familiar with. Sections 2.1 and 2.2 should mostly be review.

2.1 Sets

A **set** is a collection of objects called **elements**. If A is a set and x is an element of A, we write $\boxed{x \in A}$. Otherwise, we write $\boxed{x \notin A}$. The set containing no elements is called the **empty set**, and is denoted by the symbol $\boxed{\emptyset}$. Any set that contains at least one element is referred to as a **nonempty set**.

If we think of a set as a box potentially containing some stuff, then the empty set is a box with nothing in it. One assumption we will make is that for any set A, $A \notin A$. The language associated to sets is specific. We will often define sets using the following notation, called **set-builder notation**:

$$\boxed{S = \{x \in A \mid x \text{ satisfies some condition}\}}$$

The first part "$x \in A$" denotes what type of x is being considered. The statements to the right of the vertical bar (not to be confused with "divides") are the conditions that x must satisfy in order to be members of the set. This notation is read as "The set of all x in A such that x satisfies some condition," where "some condition" is something specific about the restrictions on x relative to A.

There are a few sets that are commonly discussed in mathematics and have predefined symbols to denote them. We've already encountered the integers, natural numbers, and real numbers. Notice that our definition of the rational numbers uses set-builder notation.

- **Natural numbers:** $\boxed{\mathbb{N} := \{1, 2, 3, \ldots\}}$. Some books will include zero in the set of natural numbers, but we do not.
- **Integers:** $\boxed{\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \ldots\}}$.
- **Rational Numbers:** $\boxed{\mathbb{Q} := \{a/b \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}}$.

- **Real Numbers:** $\boxed{\mathbb{R}}$ denotes the set of real numbers.

Since the set of natural numbers consists of the positive integers, the natural numbers are sometimes denoted by $\mathbb{Z}^+$.

If A and B are sets, then we say that A is a **subset** of B, written $\boxed{A \subseteq B}$, provided that every element of A is an element of B. Observe that $A \subseteq B$ is equivalent to "For all x in the universe of discourse, if $x \in A$, then $x \in B$."

Every nonempty set always has two rather boring subsets.

Problem 2.1. Let A be a set. Write a short proof for each of the following.

(a) $A \subseteq A$

(b) $\emptyset \subseteq A$

The next problem shows that "$\subseteq$" is a transitive relation.

Problem 2.2 (Transitivity of subsets)**.** Prove that if A, B, and C are sets such that $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Let A and B be sets in some universe of discourse U. We define the following.

- The sets A and B are **equal**, denoted $\boxed{A = B}$, if and only if $A \subseteq B$ and $B \subseteq A$. Note that if we want to prove $A = B$, then we have to do two separate mini-proofs: one for $A \subseteq B$ and one for $B \subseteq A$. It is common to label each mini-proof with "($\subseteq$)" and "($\supseteq$)", respectively.
- If $A \subseteq B$, then A is called a **proper subset** provided that $A \neq B$. In this case, we may write $\boxed{A \subset B}$ or $\boxed{A \subsetneq B}$. *Warning:* Some books use $\subset$ to mean $\subseteq$.
- The **union** of the sets A and B is $\boxed{A \cup B := \{x \in U \mid x \in A \text{ or } x \in B\}}$.
- The **intersection** of the sets A and B is $\boxed{A \cap B := \{x \in U \mid x \in A \text{ and } x \in B\}}$.
- The **set difference** of the sets A and B is $\boxed{A \setminus B := \{x \in U \mid x \in A \text{ and } x \notin B\}}$.
- The **complement of** A (relative to U) is the set $\boxed{A^c := U \setminus A = \{x \in U \mid x \notin A\}}$.
- If $A \cap B = \emptyset$, then we say that A and B are **disjoint** sets.

Example 2.3. The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of **irrational numbers**.

Problem 2.4. Prove that if A and B are sets such that $A \subseteq B$, then $B^c \subseteq A^c$.

Problem 2.5. Prove that if A and B are sets, then $A \setminus B = A \cap B^c$.

Problem 2.6. Give an example where $A \neq B$ but $A \setminus B = \emptyset$.

Consider the following collection of sets:

$$\{a\}, \{a,b\}, \{a,b,c\}, \ldots, \{a,b,c,\ldots,z\}$$

This collection has a natural way for us to "index" the sets:

$$A_1 = \{a\}, A_2 = \{a,b\}, A_3 = \{a,b,c\}, \ldots, A_{26} = \{a,b,c,\ldots,z\}$$

In this case the sets are **indexed** by the set $\{1,2,\ldots,26\}$, where the subscripts are taken from the **index set**. If we wanted to talk about an arbitrary set from this indexed collection, we could use the notation A_n.

Using indexing sets in mathematics is an extremely useful notational tool, but it is important to keep straight the difference between the sets that are being indexed, the elements in each set being indexed, the indexing set, and the elements of the indexing set.

Any set (finite or infinite) can be used as an indexing set. Often capital Greek letters are used to denote arbitrary indexing sets and small Greek letters to represent elements of these sets. If the indexing set is a subset of $\mathbb{R}$, then it is common to use Roman letters as individual indices. Of course, these are merely conventions, not rules.

- If Δ is a set and we have a collection of sets indexed by Δ, then we may write $\{S_\alpha\}_{\alpha\in\Delta}$ to refer to this collection. We read this as "the set of S-alphas over alpha in Delta."
- If a collection of sets is indexed by $\mathbb{N}$, then we may write $\{U_n\}_{n\in\mathbb{N}}$ or $\{U_n\}_{n=1}^{\infty}$.
- Borrowing from this idea, a collection $\{A_1,\ldots,A_{26}\}$ may be written as $\{A_n\}_{n=1}^{26}$.

Suppose we have a collection $\{A_\alpha\}_{\alpha\in\Delta}$.

- The **union of the entire collection** is defined via

$$\boxed{\bigcup_{\alpha\in\Delta} A_\alpha = \{x \mid x \in A_\alpha \text{ for some } \alpha \in \Delta\}.}$$

- The **intersection of the entire collection** is defined via

$$\boxed{\bigcap_{\alpha\in\Delta} A_\alpha = \{x \mid x \in A_\alpha \text{ for all } \alpha \in \Delta\}.}$$

In the special case that $\Delta = \mathbb{N}$, we write

$$\bigcup_{n=1}^{\infty} A_n = \{x \mid x \in A_n \text{ for some } n \in \mathbb{N}\} = A_1 \cup A_2 \cup A_3 \cup \cdots$$

and

$$\bigcap_{n=1}^{\infty} A_n = \{x \mid x \in A_n \text{ for all } n \in \mathbb{N}\} = A_1 \cap A_2 \cap A_3 \cap \cdots$$

Similarly, if $\Delta = \{1,2,3,4\}$, then

$$\bigcup_{n=1}^{4} A_n = A_1 \cup A_2 \cup A_3 \cup A_4 \quad \text{and} \quad \bigcap_{n=1}^{4} A_n = A_1 \cap A_2 \cap A_3 \cap A_4.$$

Notice the difference between "$\bigcup$" and "$\cup$" (respectively, "$\bigcap$" and "$\cap$").

Problem 2.7. Let $\{A_n\}_{n=1}^{26}$ be the collection from the discussion below Problem 2.6. Find each of the following.

(a) $\displaystyle\bigcup_{n=1}^{26} A_n$

(b) $\displaystyle\bigcap_{n=1}^{26} A_n$

Problem 2.8. For each $r \in \mathbb{Q}$ (the rational numbers), let N_r be the set containing all real numbers *except* r. Find each of the following.

(a) $\displaystyle\bigcup_{r\in\mathbb{Q}} N_r$

(b) $\displaystyle\bigcap_{r\in\mathbb{Q}} N_r$

A collection of sets $\{A_\alpha\}_{\alpha\in\Delta}$ is **pairwise disjoint** if $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta$.

Problem 2.9. Draw a Venn diagram of a collection of three sets that are pairwise disjoint.

Problem 2.10. Provide an example of a collection of three sets, say $\{A_1, A_2, A_3\}$, such that the collection is *not* pairwise disjoint, but $\bigcap_{n=1}^{3} A_n = \emptyset$.

Problem 2.11. Find a collection of nonempty sets $S_i \subseteq \mathbb{N}$ indexed by $i \in \mathbb{N}$ such that $S_{i+1} \subsetneq S_i$ and $\bigcap_{i=1}^{\infty} S_i = \emptyset$.

Problem 2.12. Find a collection of nonempty sets $S_i \subseteq \mathbb{N}$ indexed by $i \in \mathbb{N}$ such that $S_i \subsetneq S_{i+1}$ but $\bigcup_{i=1}^{\infty} S_i \neq \mathbb{N}$.

Problem 2.13 (DeMorgan's Law)**.** Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a collection of sets. Prove one of the following.

(a) $\displaystyle\left(\bigcup_{\alpha\in\Delta} A_\alpha\right)^C = \bigcap_{\alpha\in\Delta} A_\alpha^C$

(b) $\displaystyle\left(\bigcap_{\alpha\in\Delta} A_\alpha\right)^C = \bigcup_{\alpha\in\Delta} A_\alpha^C$

Problem 2.14 (Distribution of Union and Intersection)**.** Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a collection of sets and let B be any set. Prove one of the following.

(a) $\displaystyle B \cup \left(\bigcap_{\alpha\in\Delta} A_\alpha\right) = \bigcap_{\alpha\in\Delta} (B \cup A_\alpha)$

(b) $\displaystyle B \cap \left(\bigcup_{\alpha\in\Delta} A_\alpha\right) = \bigcup_{\alpha\in\Delta} (B \cap A_\alpha)$

For each $n \in \mathbb{N}$, we define an n-**tuple** to be an ordered list of n elements of the form $(a_1, a_2, \ldots, a_n)$. We refer to a_i as the ith **component** (or **coordinate**) of $(a_1, a_2, \ldots, a_n)$. Two n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are equal if and only if $a_i = b_i$ for all $1 \leq i \leq n$. A 2-tuple (a, b) is more commonly referred to as an **ordered pair** while a 3-tuple (a, b, c) is often called an **ordered triple**.

We can use the notion of n-tuples to construct new sets from existing sets. If A and B are sets, the **Cartesian product** (or **direct product**) of A and B, denoted $A \times B$ (read as "A times B" or "A cross B"), is the set of all ordered pairs where the first component is from A and the second component is from B. In set-builder notation, we have

$$\boxed{A \times B := \{(a,b) \mid a \in A, b \in B\}}.$$

We similarly define the Cartesian product of n sets, say $A_1, \ldots, A_n$, by

$$\boxed{\prod_{i=1}^{n} A_i := A_1 \times \cdots \times A_n := \{(a_1, \ldots, a_n) \mid a_j \in A_j \text{ for all } 1 \leq j \leq n\}},$$

where A_i is referred to as the ith **factor** of the Cartesian product. As a special case, the set

$$\underbrace{A \times \cdots \times A}_{n \text{ factors}}$$

is often abbreviated as A^n.

Example 2.15. The standard two-dimensional plane $\mathbb{R}^2$ and standard three space $\mathbb{R}^3$ are familiar examples of Cartesian products. In particular, we have

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x, y \in \mathbb{R}\}$$

and

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}.$$

Problem 2.16. If A is a set, then what is $A \times \emptyset$ equal to?

Problem 2.17. Given sets A and B, when will $A \times B$ be equal to $B \times A$?

We now turn our attention to subsets of Cartesian products.

Problem 2.18. Prove that if A, B, C, and D are sets such that $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

Problem 2.19. Is it true that if $A \times B \subseteq C \times D$, then $A \subseteq C$ and $B \subseteq D$? Don't forget to think about cases involving the empty set.

Problem 2.20. Is every subset of $C \times D$ of the form $A \times B$, where $A \subseteq C$ and $B \subseteq D$? If so, prove it. If not, find a counterexample.

Problem 2.21. If A, B, and C are nonempty sets, is $A \times B$ a subset of $A \times B \times C$?

Problem 2.22. Let A, B, C, and D be sets. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

(b) $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$

(c) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(d) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(e) $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$

2.2 Functions

Let A and B be sets. A **relation** R **from** A **to** B is a subset of $A \times B$. If R is a relation from A to B and $(a,b) \in R$, then we say that a **is related to** b and we may write aRb in place of $(a,b) \in R$.

A function is a special type of relation, where the basic building blocks are a first set and a second set, say X and Y, and a "correspondence" that assigns *every* element of X to *exactly one* element of Y. More formally, if X and Y are nonempty sets, a **function** f **from** X **to** Y is a relation from X to Y such that for every $x \in X$, there exists a unique $y \in Y$ such that $(x,y) \in f$. The set X is called the **domain** of f and is denoted by $\boxed{\mathrm{Dom}(f)}$. The set Y is called the **codomain** of f and is denoted by $\boxed{\mathrm{Codom}(f)}$ while the subset of the codomain defined via

$$\boxed{\mathrm{Rng}(f) := \{y \in Y \mid \text{there exists } x \text{ such that } (x,y) \in f\}}$$

is called the **range** of f or the **image of** X under f.

There is a variety of notation and terminology associated to functions. We will write $\boxed{f : X \to Y}$ to indicate that f is a function from X to Y. We will make use of statements such as "Let $f : X \to Y$ be the function defined via..." or "Define $f : X \to Y$ via...", where f is understood to be a function in the second statement. Sometimes the word **mapping** (or **map**) is used in place of the word function. If $(a,b) \in f$ for a function f, we often write $\boxed{f(a) = b}$ and say that "f maps a to b" or "f of a equals b". In this case, a may be called an **input** of f and is the **preimage** of b under f while b is called an **output** of f and is the **image** of a under f. Note that the domain of a function is the set of inputs while the range is the set of outputs for the function.

Notice that we can interpret our definition of function in terms of existence and uniqueness. That is, $f : X \to Y$ is a function provided:

1. (Existence) For each $x \in X$, there exists $y \in Y$ such that $y = f(x)$, and
2. (Uniqueness) If $f(x) = y_1$ and $f(x) = y_2$, then $y_1 = y_2$.

In other words, every element of the domain is utilized and is utilized exactly once. However, there are no restrictions on whether an element of the codomain ever appears in the second coordinate of an ordered pair in the relation. Yet if an element of Y is in the range of f, it may appear in more than one ordered pair in the relation.

It follows immediately from the definition of function that two functions are equal if and only if they have the same domain, same codomain, and the same set of ordered pairs in the relation. That is, functions f and g are equal if and only if $\mathrm{Dom}(f) = \mathrm{Dom}(g)$, $\mathrm{Codom}(f) = \mathrm{Codom}(g)$, and $f(x) = g(x)$ for all $x \in X$.

Since functions are special types of relations, we can represent them using digraphs and graphs when practical. Digraphs for functions are often called **function** (or **mapping**) **diagrams**. When drawing function diagrams, it is standard practice to put the vertices for the domain on the left and the vertices for the codomain on the right, so that all directed edges point from left to right. We may also draw an additional arrow labeled by the name of the function from the domain to the codomain.

Example 2.23. Let $X = \{a,b,c,d\}$ to $Y = \{1,2,3,4\}$ and define the relation f from X to Y via

$$f = \{(a,2),(b,4),(c,4),(d,1)\}.$$

Since each element X appears exactly once as a first coordinate, f is a function with domain X and codomain Y (i.e., $f : X \to Y$). In this case, we see that $\mathrm{Rng}(f) = \{1,2,4\}$. Moreover, we can write things like $f(a) = 2$ and $c \mapsto 4$, and say things like "f maps b to 4" and "the image of d is 1." The function diagram for f is depicted in Figure 2.1.

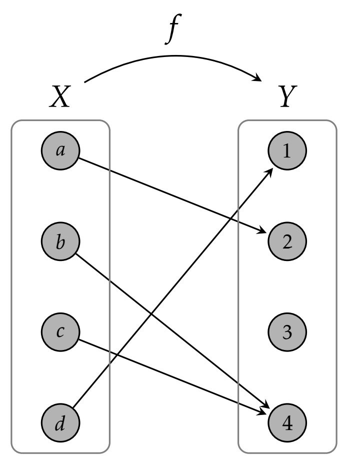

Figure 2.1: Function diagram for a function from $X = \{a,b,c,d,\}$ to $Y = \{1,2,3,4\}$.

Problem 2.24. What properties does the digraph for a relation from X to Y need to have in order for it to represent a function?

Problem 2.25. In high school I am sure that you were told that a graph represents a function if it passes the **vertical line test**. Carefully state what the vertical line test says and then explain why it works.

Sometimes we can define a function using a formula. For example, we can write $f(x) = x^2 - 1$ to mean that each x in the domain of f maps to $x^2 - 1$ in the codomain. However, notice that providing only a formula is ambiguous! A function is determined by its domain, codomain, and the correspondence between these two sets. If we only provide a description for the correspondence, it is not clear what the domain and codomain are. Two functions that are defined by the same formula, but have different domains or codomains are *not* equal.

Example 2.26. The function $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x^2 - 1$ is not equal to the function $g : \mathbb{N} \to \mathbb{R}$ defined by $g(x) = x^2 - 1$ since the two functions do not have the same domain.

Sometimes we rely on context to interpret the domain and codomain. For example, in a calculus class, when we describe a function in terms of a formula, we are implicitly assuming that the domain is the largest allowable subset of $\mathbb{R}$—sometimes called the **default domain**—that makes sense for the given formula while the codomain is $\mathbb{R}$.

Example 2.27. If we write $f(x) = x^2 - 1$, $g(x) = \sqrt{x}$, and $h(x) = \frac{1}{x}$ without mentioning the domains, we would typically interpret these as the functions $f : \mathbb{R} \to \mathbb{R}$, $g : [0,\infty) \to \mathbb{R}$, and $h : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ that are determined by their respective formulas.

Problem 2.28. Provide an example of each of the following. You may draw a function diagram, write down a list of ordered pairs, or a write a formula as long as the domain and codomain are clear.

(a) A function f from a set with 4 elements to a set with 3 elements such that $\operatorname{Rng}(f) = \operatorname{Codom}(f)$.

(b) A function g from a set with 4 elements to a set with 3 elements such that $\operatorname{Rng}(g)$ is strictly smaller than $\operatorname{Codom}(g)$.

There are a few special functions that we should know the names of. Let X and Y be nonempty sets.

- If $X \subseteq Y$, then the function $\iota : X \to Y$ defined via $\iota(x) = x$ is called the **inclusion map from X into Y**. Note that "ι" is the Greek letter "iota".
- If the domain and codomain are equal, the inclusion map has a special name. If X is a nonempty set, then the function $i_X : X \to X$ defined via $i_X(x) = x$ is called the **identity map** (or **identity function**) **on** X.
- Any function $f : X \to Y$ defined via $f(x) = c$ for a fixed $c \in Y$ is called a **constant function**.
- A **piecewise-defined function** (or **piecewise function**) is a function defined by specifying its output on a partition of the domain. Note that "piecewise" is a way of expressing the function, rather than a property of the function itself.

Example 2.29. The function $f : \mathbb{R} \to \mathbb{R}$ defined via

$$f(x) = \begin{cases} x^2 - 1, & \text{if } x \text{ is negative,} \\ 17, & \text{if } x = 0, \\ -x, & \text{if } x \text{ is positive} \end{cases}$$

is an example of a piecewise-defined function.

It is important to point out that not every function can be described using a formula! Despite your prior experience, functions that can be represented succinctly using a formula are rare.

The next problem illustrates that some care must be taken when attempting to define a function.

Problem 2.30. For each of the following, explain why the given description does not define a function.

(a) Define $f : \{1,2,3\} \to \{1,2,3\}$ via $f(a) = a - 1$.

(b) Define $g : \mathbb{N} \to \mathbb{Q}$ via $g(n) = \frac{n}{n-1}$.

(c) Let $A_1 = \{1,2,3\}$ and $A_2 = \{3,4,5\}$. Define $h : A_1 \cup A_2 \to \{1,2\}$ via

$$h(x) = \begin{cases} 1, & \text{if } x \in A_1 \\ 2, & \text{if } x \in A_2. \end{cases}$$

(d) Define $s : \mathbb{Q} \to \mathbb{Z}$ via $s(a/b) = a + b$.

In mathematics, we say that an expression is **well defined** (or **unambiguous**) if its definition yields a unique interpretation. Otherwise, we say that the expression is not well defined (or is **ambiguous**). For example, if $a, b, c \in \mathbb{R}$, then the expression abc is well defined since it does not matter if we interpret this as $(ab)c$ or $a(bc)$ since the real numbers are associative under multiplication.

When we attempt to define a function, it may not be clear without doing some work that our definition really does yield a function. If there is some potential ambiguity in the definition of a function that ends up not causing any issues, we say that the function is well defined. However, this phrase is a bit of misnomer since all functions are well defined. The issue of whether a description for a proposed function is well defined often arises when defining things in terms of representatives of equivalence classes, or more generally in terms of how an element of the domain is written. For example, the descriptions given in parts (c) and (d) of Problem 2.30 are not well defined. To show that a potentially ambiguous description for a function $f : X \to Y$ is well defined prove that if a and b are two representations for the same element in X, then $f(a) = f(b)$.

Let $f : X \to Y$ be a function.

- The function f is said to be **injective** (or **one-to-one**) if for all $y \in \text{Rng}(f)$, there is a unique $x \in X$ such that $y = f(x)$.
- The function f is said to be **surjective** (or **onto**) if for all $y \in Y$, there exists $x \in X$ such that $y = f(x)$.
- If f is both injective and surjective, we say that f is **bijective**.

An injective function is also called an **injection**, a surjective function is called a **surjection**, and a bijective function is called a **bijection** (or a **one-to-one correspondence**). A one-to-one correspondence should not be confused with a one-to-one function which may not be surjective. To prove that a function $f : X \to Y$ is an injection, we must prove that if $f(x_1) = f(x_2)$, then $x_1 = x_2$. To show that f is surjective, you should start with an arbitrary $y \in Y$ and then work to show that there exists $x \in X$ such that $y = f(x)$.

Problem 2.31. Assume that X and Y are finite sets. Provide an example of each of the following. You may draw a function diagram, write down a list of ordered pairs, or a write a formula as long as the domain and codomain are clear.

(a) A function $f : X \to Y$ that is injective but not surjective.

(b) A function $f : X \to Y$ that is surjective but not injective.

(c) A function $f : X \to Y$ that is a bijection.

(d) A function $f : X \to Y$ that is neither injective nor surjective.

Problem 2.32. Provide an example of each of the following. You may either draw a graph or write down a formula. Make sure you have the correct domain.

(a) A function $f : \mathbb{R} \to \mathbb{R}$ that is injective but not surjective.

(b) A function $f : \mathbb{R} \to \mathbb{R}$ that is surjective but not injective.

(c) A function $f : \mathbb{R} \to \mathbb{R}$ that is a bijection.

(d) A function $f : \mathbb{R} \to \mathbb{R}$ that is neither injective nor surjective.

(e) A function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ that is injective.

Problem 2.33. Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. Fill in the blank with the appropriate word.

> The function $f : X \to \mathbb{R}$ is _________ if and only if every horizontal line hits the graph of f *at most once*.

This statement is often called the **horizontal line test**. Explain why the horizontal line test is true.

Problem 2.34. Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. Fill in the blank with the appropriate word.

> The function $f : X \to \mathbb{R}$ is _________ if and only if every horizontal line hits the graph of f *at least once*.

Explain why this statement is true.

Problem 2.35. Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. Fill in the blank with the appropriate word.

> The function $f : X \to \mathbb{R}$ is _________ if and only if every horizontal line hits the graph of f *exactly once*.

Explain why this statement is true.

Problem 2.36. Determine whether each of the following functions is injective, surjective, both, or neither. In each case, you should provide a proof or a counterexample as appropriate.

(a) Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$

(b) Define $g : \mathbb{R} \to [0, \infty)$ via $g(x) = x^2$

(c) Define $h : \mathbb{R} \to \mathbb{R}$ via $h(x) = x^3$

(d) Define $k : \mathbb{R} \to \mathbb{R}$ via $k(x) = x^3 - x$

(e) Define $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ via $c(x, y) = x^2 + y^2$

(f) Define $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ via $f(n) = (n, n)$

(g) Define $g : \mathbb{Z} \to \mathbb{Z}$ via

$$g(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

(h) Define $\ell : \mathbb{Z} \to \mathbb{N}$ via

$$\ell(n) = \begin{cases} 2n + 1, & \text{if } n \geq 0 \\ -2n, & \text{if } n < 0 \end{cases}$$

The next two results should not come as as surprise.

Problem 2.37. Prove that the inclusion map $\iota : X \to Y$ for $X \subseteq Y$ is an injection.

Problem 2.38. Prove that the identity function $i_X : X \to X$ is a bijection.

If $f : X \to Y$ and $g : Y \to Z$ are functions, we define $g \circ f : X \to Z$ via $\boxed{(g \circ f)(x) = g(f(x))}$. The function $g \circ f$ is called the **composition of** f **and** g. It is important to notice that the function on the right is the one that "goes first." Moreover, we cannot compose any two random functions since the codomain of the first function must agree with the domain of the second function. In particular, $f \circ g$ may not be a sensible function even when $g \circ f$ exists. Figure 2.2 provides a visual representation of function composition in terms of function diagrams.

Example 2.39. Consider the inclusion map $\iota : X \to Y$ such that X is a proper subset of Y and suppose $f : Y \to Z$ is a function. Then the composite function $f \circ \iota : X \to Z$ is given by

$$f \circ \iota(x) = f(\iota(x)) = f(x)$$

for all $x \in X$. Notice that $f \circ \iota$ is simply the function f but with a smaller domain. In this case, we say that $f \circ \iota$ is the **restriction of** f **to** X, which is often denoted by $\boxed{f|_X}$.

The next problem illustrates that $f \circ g$ and $g \circ f$ need not be equal even when both composite functions exist.

Problem 2.40. Define $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$ and $g(x) = 3x - 5$, respectively. Determine formulas for the composite functions $f \circ g$ and $g \circ f$.

The next problem tells us that function composition is associative.

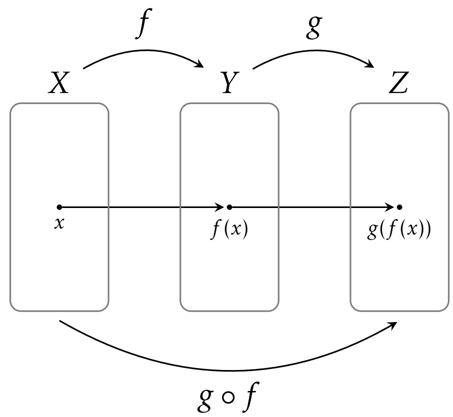

Figure 2.2: Visual representation of function composition.

Problem 2.41. Prove that if $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$ are functions, then $(h \circ g) \circ f = h \circ (g \circ f)$.

Problem 2.42. In each case, give examples of finite sets X, Y, and Z, and functions $f : X \to Y$ and $g : Y \to Z$ that satisfy the given conditions. Drawing a function diagram is sufficient.

(a) f is surjective, but $g \circ f$ is not surjective.

(b) g is surjective, but $g \circ f$ is not surjective.

(c) f is injective, but $g \circ f$ is not injective.

(d) g is injective, but $g \circ f$ is not injective.

Problem 2.43. Prove that if $f : X \to Y$ and $g : Y \to Z$ are both surjective functions, then $g \circ f$ is also surjective.

Problem 2.44. Prove that if $f : X \to Y$ and $g : Y \to Z$ are both injective functions, then $g \circ f$ is also injective.

Problem 2.45. Prove that if $f : X \to Y$ and $g : Y \to Z$ are both bijections, then $g \circ f$ is also a bijection.

Problem 2.46. Assume that $f : X \to Y$ and $g : Y \to Z$ are both functions. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) If $g \circ f$ is injective, then f is injective.

(b) If $g \circ f$ is injective, then g is injective.

(c) If $g \circ f$ is surjective, then f is surjective.

(d) If $g \circ f$ is surjective, then g is surjective.

There are two important types of sets related to functions. Let $f : X \to Y$ be a function.

- If $S \subseteq X$, the **image** of S under f is defined via

$$\boxed{f(S) := \{f(x) \mid x \in S\}}.$$

- If $T \subseteq Y$, the **preimage** (or **inverse image**) of T under f is defined via

$$\boxed{f^{-1}(T) := \{x \in X \mid f(x) \in T\}}.$$

The image of a subset S of the domain is simply the subset of the codomain we obtain by mapping the elements of S. It is important to emphasize that the function f maps *elements* of X to *elements* of Y, but we can apply f to a *subset* of X to yield a *subset* of Y. That is, if $S \subseteq X$, then $f(S) \subseteq Y$. Note that the image of the domain is the same as the range of the function. That is, $f(X) = \operatorname{Rng}(f)$.

When it comes to preimages, the notation $f^{-1}(T)$ should not be confused with an inverse function (which may or may not exist for an arbitrary function f). For $T \subseteq Y$, $f^{-1}(T)$ is the set of elements in the domain that map to elements in T. As a special case, $f^{-1}(\{y\})$ is the set of elements in the domain that map to $y \in Y$. If $y \notin \operatorname{Rng}(f)$, then $f^{-1}(\{y\}) = \emptyset$. Notice that if $y \in Y$, $f^{-1}(\{y\})$ is always a sensible thing to write while $f^{-1}(y)$ only makes sense if f^{-1} is a function. Also, note that the preimage of the codomain is the domain. That is, $f^{-1}(Y) = X$.

Problem 2.47. Define $f : \mathbb{Z} \to \mathbb{Z}$ via $f(x) = x^2$. List elements in each of the following sets.

(a) $f(\{0, 1, 2\})$

(b) $f^{-1}(\{0, 1, 4\})$

Problem 2.48. Find functions f and g and sets S and T such that $f(f^{-1}(T)) \neq T$ and $g^{-1}(g(S)) \neq S$.

Problem 2.49. Suppose $f : X \to Y$ is an injection and A and B are disjoint subsets of X. Are $f(A)$ and $f(B)$ necessarily disjoint subsets of Y? If so, prove it. Otherwise, provide a counterexample.

Problem 2.50. Let $f : X \to Y$ be a function and suppose $A, B \subseteq X$ and $C, D \subseteq Y$. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) If $A \subseteq B$, then $f(A) \subseteq f(B)$.

(b) If $C \subseteq D$, then $f^{-1}(C) \subseteq f^{-1}(D)$.

(c) $f(A \cup B) \subseteq f(A) \cup f(B)$.

(d) $f(A \cup B) \supseteq f(A) \cup f(B)$.

(e) $f(A \cap B) \subseteq f(A) \cap f(B)$.

(f) $f(A \cap B) \supseteq f(A) \cap f(B)$.

(g) $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$.

(h) $f^{-1}(C \cup D) \supseteq f^{-1}(C) \cup f^{-1}(D)$.

(i) $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$.

(j) $f^{-1}(C \cap D) \supseteq f^{-1}(C) \cap f^{-1}(D)$.

(k) $A \subseteq f^{-1}(f(A))$.

(l) $A \supseteq f^{-1}(f(A))$.

(m) $f(f^{-1}(C)) \subseteq C$.

(n) $f(f^{-1}(C)) \supseteq C$.

2.3 The Real Numbers

The real numbers form the foundation of mathematical analysis. It is worth pointing out that one can carefully construct the real numbers from the natural numbers. However, that will not be the approach we take. Instead, we will simply list the axioms that the real numbers satisfy. Recall that an axiom is a statement that is assumed to be true without proof. These are the basic building blocks from which all theorems are proved. Our axioms for the real numbers fall into three categories:

1. **Field Axioms:** These axioms provide the essential properties of arithmetic involving addition and subtraction.

2. **Order Axioms:** These axioms provide the necessary properties of inequalities.

3. **Completeness Axiom:** This axiom guarantees that the familiar number line representing the real numbers does not have any "gaps". We will not introduce this axiom until Chapter 3.

Field Axioms 2.51. There exist functions $(a,b) \mapsto a+b$ and $(a,b) \mapsto ab$ from $\mathbb{R}^2$ to $\mathbb{R}$ satisfying:

(F1) (Associativity for Addition) For all $a,b,c \in \mathbb{R}$ we have $(a+b)+c = a+(b+c)$;

(F2) (Commutativity for Addition) For all $a,b \in \mathbb{R}$, we have $a+b = b+a$;

(F3) (Additive Identity) There exists $0 \in \mathbb{R}$ such that for all $a \in \mathbb{R}$, $0+a = a$;

(F4) (Additive Inverses) For all $a \in \mathbb{R}$ there exists $-a \in \mathbb{R}$ such that $a+(-a) = 0$;

(F5) (Associativity for Multiplication) For all $a,b,c \in \mathbb{R}$ we have $(ab)c = a(bc)$;

(F6) (Commutativity for Multiplication) For all $a,b \in \mathbb{R}$, we have $ab = ba$;

(F7) (Multiplicative Identity) There exists $1 \in \mathbb{R}$ such that $1 \neq 0$ and for all $a \in \mathbb{R}$, $1a = a$;

(F8) (Multiplicative Inverses) For all $a \in \mathbb{R} \setminus \{0\}$ there exists $a^{-1} \in \mathbb{R}$ such that $aa^{-1} = 1$.

(F9) (Distributive Property) For all $a,b,c \in \mathbb{R}$, $a(b+c) = ab + ac$;

In the language of abstract algebra, Axioms (F1)–(F4) and (F5)–(F8) make each of $\mathbb{R}$ and $\mathbb{R} \setminus \{0\}$ an abelian group under addition and multiplication, respectively. Axiom (F9) provides a way for the operations of addition and multiplication to interact. Collectively, Axioms (F1)–(F9) make the real numbers a **field**. It follows from the axioms that the elements 0 and 1 of $\mathbb{R}$ are the unique additive and multiplicative identities. For every $a \in \mathbb{R}$, the elements $-a$ and a^{-1} (as long as $a \neq 0$) are also the unique additive and multiplicative inverses. We will take these facts for granted. For every $a,b \in \mathbb{R}$ and $n \in \mathbb{Z}$, we define the following:

- $a - b := a + (-b)$
- $\dfrac{a}{b} := ab^{-1}$ (for $b \neq 0$)
- $a^n := \begin{cases} \overbrace{aa\cdots a}^{n}, & \text{if } n \in \mathbb{N} \\ 1, & \text{if } n = 0 \text{ and } a \neq 0 \\ \dfrac{1}{a^{-n}}, & \text{if } -n \in \mathbb{N} \text{ and } a \neq 0 \end{cases}$

Using the Field Axioms, we could prove each of the statements in the following theorem. However, we will take each for granted.

Theorem 2.52. For all $a,b,c \in \mathbb{R}$, we have the following:

(a) $a = b$ if and only if $a + c = b + c$;

(b) $0a = 0$;

(c) $-a = (-1)a$;

(d) $(-1)^2 = 1$;

(e) $-(-a) = a$;

(f) If $a \neq 0$, then $(a^{-1})^{-1} = a$;

(g) If $a \neq 0$ and $ab = ac$, then $b = c$.

(h) If $ab = 0$, then either $a = 0$ or $b = 0$.

Problem 2.53. Carefully prove that for all $a,b \in \mathbb{R}$, we have $(a+b)(a-b) = a^2 - b^2$.

Order Axioms 2.54. For $a, b, c \in \mathbb{R}$, there is a relation $<$ on $\mathbb{R}$ satisfying:

(O1) (Trichotomy Law) If $a \neq b$, then either $a < b$ or $b < a$ but not both;

(O2) (Transitivity) If $a < b$ and $b < c$, then $a < c$;

(O3) If $a < b$, then $a + c < b + c$;

(O4) If $a < b$ and $0 < c$, then $ac < bc$;

Given Axioms (O1)–(O4) above, we say that the real numbers are **linearly ordered** (or **totally ordered**). We call numbers greater than zero **positive** and those greater than or equal to zero **nonnegative**. There are similar definitions for **negative** and **nonpositive**. For $a, b \in \mathbb{R}$, we define:

- $a > b$ if $b < a$;
- $a \leq b$ if $a < b$ or $a = b$;
- $a \geq b$ if $b \leq a$.

Using the Order Axioms, we can prove many familiar facts.

Problem 2.55. Prove that for all $a, b \in \mathbb{R}$, if $a, b > 0$, then $a + b > 0$, and if $a, b < 0$, then $a + b < 0$.

The next problem extends Axiom (O3).

Problem 2.56. Prove that for all $a, b, c, d \in \mathbb{R}$, if $a < b$ and $c < d$, then $a + c < b + d$.

Problem 2.57. For all $a \in \mathbb{R}$, $a > 0$ if and only if $-a < 0$.

Problem 2.58. Prove that if a, b, c, and d are positive real numbers such that $a < b$ and $c < d$, then $ac < bd$.

We will take the following theorem for granted. Both statements can be proved using the axioms above.

Theorem 2.59. For all $a, b \in \mathbb{R}$, we have the following:

(a) $ab > 0$ if and only if either $a, b > 0$ or $a, b < 0$;

(b) $ab < 0$ if and only if $a < 0 < b$ or $b < 0 < a$.

Problem 2.60. Prove that for all positive real numbers a and b, $a < b$ if an only if $a^2 < b^2$.

Consider using three cases when approaching the following problem.

Problem 2.61. Prove that for all $a \in \mathbb{R}$, we have $a^2 \geq 0$.

It might come as a surprise that the following result requires proof.

Problem 2.62. Prove that $0 < 1$.

The previous problem together with Problem 2.57 implies that $-1 < 0$ as you expect. It also follows from Axiom (O3) that for all $n \in \mathbb{Z}$, we have $n < n+1$. We assume that there are no integers between n and $n+1$.

Problem 2.63. Prove that for all $a \in \mathbb{R}$, if $a > 0$, then $a^{-1} > 0$, and if $a < 0$, then $a^{-1} < 0$.

Problem 2.64. Prove that for all $a, b \in \mathbb{R}$, if $a < b$, then $-b < -a$.

The last few results allow us to take for granted our usual understanding of which real numbers are positive and which are negative. The next problem yields a result that extends the previous problem.

Problem 2.65. Prove that for all $a, b, c \in \mathbb{R}$, if $a < b$ and $c < 0$, then $bc < ac$.

We could spend weeks building up from the axioms all of the machinery necessary for the rest of the course. Instead we will toss in a few additional axioms to save ourselves a little time.

Additional Order Axioms 2.66. The real numbers satisfy each of the following:

(O5) For every $x \in \mathbb{R}$, there exists $a, b \in \mathbb{R}$ such that $a < x < b$;

(O6) For every $a, b \in \mathbb{R}$, if $a < b$, there exists $x \in \mathbb{R}$ such that $a < x < b$ (in particular, $\frac{a+b}{2}$ is between a and b);

(O7) For every $a \in \mathbb{R}$, there exists $n \in \mathbb{Z}$ such that $n \leq a < n+1$.

Axiom (O7) is sometimes referred to as the **Archimedean Principle**. It turns out that we could derive this axiom from the **Completeness Axiom**, which we will introduce in the next chapter.

Problem 2.67. Prove that for any positive real number a, there exists $N \in \mathbb{N}$ such that $0 < \frac{1}{N} < a$.

For $a, b \in \mathbb{R}$ with $a < b$, we define the following **intervals**:

- $\boxed{(a, b) := \{x \in \mathbb{R} \mid a < x < b\}}$
- $\boxed{(a, \infty) := \{x \in \mathbb{R} \mid a < x\}}$
- $\boxed{(-\infty, b) := \{x \in \mathbb{R} \mid x < b\}}$
- $\boxed{[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}}$

We analogously define $[a, b)$, $(a, b]$, $[a, \infty)$, and $(-\infty, b]$. Intervals of the form (a, b), $(-\infty, b)$, and (a, ∞) are called **open intervals** while $[a, b]$, $(-\infty, b]$, and $[a, \infty)$ are referred to as **closed intervals**. A **finite length interval** is any interval of the form (a, b), $[a, b)$, $(a, b]$, and $[a, b]$. For finite length intervals, a and b are called the **endpoints** of the interval.

Notice that Axiom (O5) says that every real number is contained in a finite open interval. In particular, Axiom (O7) says that every non-integer is contained in an open interval with consecutive integer endpoints. Axiom (O6) tells us that every open interval is nonempty. In fact, repeated applications of Axiom (O6) implies that every open interval contains infinitely many points.

Problem 2.68. Assume that there is a positive element of the preimage of $\{2\}$ under the function $f(x) = x^2$ from the reals to the reals. That is, assume $\sqrt{2}$ exists. Prove $\sqrt{2} \in (1, 2)$.

Recall that $\sqrt{2}$ is an irrational number. The previous problem provides an example of an irrational number occurring between a pair of distinct rational numbers. The following problems are a good challenge to generalize this.

Problem 2.69. Prove that between any two distinct real numbers there is a rational number.

Problem 2.70. Prove that between any two distinct real numbers there is an irrational number.

Repeated applications of the previous two problems implies that every open interval contains infinitely many rational numbers and infinitely many irrational numbers. In light of these two problems, we say that both the rationals and irrationals are **dense** in every open interval. In particular, they are dense in the real numbers.

There is a special function that we can now introduce. Given $a \in \mathbb{R}$, we define the **absolute value of** a, denoted $|a|$, via

$$|a| := \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0. \end{cases}$$

Problem 2.71. Prove that for all $a \in \mathbb{R}$, $|a| \geq 0$ with equality only if $a = 0$.

Problem 2.72. Prove that for all $a, b \in \mathbb{R}$, if $\pm a \leq b$, then $|a| \leq b$. *Note:* Writing $\pm a \leq b$ is an abbreviation for $a \leq b$ and $-a \leq b$.

Problem 2.73. Prove that for all $a \in \mathbb{R}$, $|a|^2 = a^2$.

Problem 2.74. Prove that for all $a \in \mathbb{R}$, $\pm a \leq |a|$.

Problem 2.75. Prove that for all $a, r \in \mathbb{R}$, $|a| \leq r$ if and only if $-r \leq a \leq r$.

In the previous problem, it must be the case that r is nonnegative. The letter r was used because it is the first letter of the word "radius". If r is positive, we can think of the interval $(-r, r)$ as the interior of a 1-dimensional circle with radius r centered at 0.

Problem 2.76. Prove that for all $a, b \in \mathbb{R}$, $|ab| = |a||b|$.

Consider using Problems 2.74 and 2.75 when attacking the next problem. This result is extremely useful.

Problem 2.77 (Triangle Inequality)**.** Prove that for all $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

The next problem is related to the Triangle Inequality

Problem 2.78 (Reverse Triangle Inequality)**.** Prove that for all $a, b \in \mathbb{R}$, $|a - b| \geq ||a| - |b||$.