Homework 11

Abstract Algebra I

Complete the following problems. Note that you should only use results that we've discussed so far this semester.

Problem 1. Consider the ring $M_2(\mathbb{R})$ (i.e., the ring of 2×2 matrices with real number entries, where the operation is matrix multiplication). Recall that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(A) = ad - bc$. Is det a ring homomorphism? Justify your answer.

Problem 2. Define $\phi : \mathbb{Z}_4 \to \mathbb{Z}_{12}$ via $\phi(x) = 3x$. Is ϕ a ring homomorphism? Justify your answer.

Problem 3. Consider the ring $M_2(\mathbb{Z})$. Let $I = \left\{ \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \mid a, c \in \mathbb{Z} \right\}$. Show that I is a left ideal, but not a right ideal.

Problem 4. Let R be a ring. If there exists a positive integer n such that

$$\underbrace{a + a + \cdots + a}_{n} = 0$$

for all $a \in R$, then the least such positive integer is called the **characteristic** of R. If no such positive integer exists, then R is of characteristic 0. Find the characteristic of each of the following rings.

(a) $\mathbb{Z}_6$

(b) $\mathbb{Z}$

(c) $\mathbb{R}$

Problem 5. Prove **one** of the following.

(a) Prove that the characteristic of an integral domain is either 0 or prime.

(b) Let R be a commutative ring with prime characteristic p. Prove that if $x, y \in R$, then $(x + y)^p = x^p + y^p$.

Problem 6. Consider $E = \{0, 2, 4, 6, 8\} \subseteq \mathbb{Z}_{10}$. Find the field of fractions of E in $\mathbb{Z}_{10}$.

Problem 7. Define $\phi : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ via $\phi(x) = 6x$.

(a) Prove that ϕ is a ring homomorphism.

(b) Determine whether $\mathbb{Z}_{10}/\ker(\phi)$ is a field.

(c) Is $\ker(\phi)$ a maximal ideal of $\mathbb{Z}_{10}$?

Problem 8. A **simple ring** is a ring with no nonzero proper 2-sided ideals. If R is a ring, then the **center** of R is defined to be $Z(R) := \{x \in R \mid rx = xr \text{ for all } r \in R\}$. Prove that the center of a simple ring with 1 is a field. *Note:* You must first show that the center is a subring.

Problem 9. Let R be a ring and let I be a right ideal of R. Suppose there exists an element $a \in R$ such that $a^2 = a$ (such an element is called **idempotent**). Let $J = \{x \in I \mid ax = 0\}$. Prove that J is a right ideal of R.

Problem 10. Let $\phi : R \to S$ be a ring homomorphism, where R is a ring with 1, call it 1_R.

(a) Prove that $\phi(1_R)$ is the multiplicative identity in $\phi(R)$.

(b) Provide an example of a ring homomorphism where S has a multiplicative identity that is not equal to $\phi(1_R)$ or prove that such an example does not exist.

Problem 11. Prove **one** of the following.

(a) Let R be a commutative ring with 1. The **radical** of an ideal I in R is defined to be $\sqrt{I} := \{x \in R \mid x^n \in I \text{ for some } n \in \mathbb{Z}^+\}$. Prove that every prime ideal is radical.

(b) Let R be a commutative ring with 1 and let $U(R)$ be the group of units in R. Prove that R has a unique maximal ideal iff $R \setminus U(R)$ is an ideal. *Note:* You may use Theorem 38 from our class notes.

Problem 12. Prove **one** of the following.

(a) Prove that any subfield of $\mathbb{R}$ must contain $\mathbb{Q}$.

(b) Prove that a quotient of a principal ideal domain by a prime ideal is still a principal ideal domain.