Homework 2

Abstract Algebra I

Complete the following problems.

Problem 1. Determine whether each of the following binary operations is (i) associative and (ii) commutative.

(a) The operation $\star$ on $\mathbb{R}$ defined via $a \star b = a + b + ab$.

(b) The operation $\circ$ on $\mathbb{Q}$ defined via $a \circ b = \dfrac{a+b}{5}$.

(c) The operation $\circledcirc$ on $\mathbb{Z} \times \mathbb{Z}$ defined via $(a,b) \circledcirc (c,d) = (ad + bc, bd)$.

(d) The operation $\circledast$ on $\mathbb{Q} \setminus \{0\}$ defined via $a \circledast b = \dfrac{a}{b}$.

(e) The operation $\ominus$ on $\mathbb{R}/I := \{x \in \mathbb{R} \mid 0 \leq x < 1\}$ defined via $a \ominus b = a + b - \lfloor a + b \rfloor$ (i.e., $a \ominus b$ is the fractional part of $a + b$).

Problem 2. Determine which of the following sets are groups under the given operation. Justify your answer.

(a) $\mathbb{Z}/n\mathbb{Z}$ under multiplication mod n.

(b) Set of rational numbers in lowest terms whose denominators are odd under addition. *Note:* Since we can write $0 = 0/1$, 0 is included in this set.

(c) Set of rational numbers in lowest terms whose denominators are even together with 0 under addition.

(d) Set of rational numbers of absolute value less than 1 under addition.

(e) $\mathbb{R}/I$ under $\ominus$ as defined in Problem 1(e).

Problem 3. Let $G = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. Prove one of the following.

(a) The set G is a group under addition.

(b) If $H = G \setminus \{0\}$, then H is a group under multiplication.

Problem 4. Assume G is a group and let $x \in G$. Prove one of the following.

(a) If $a, b \in \mathbb{Z}$, then $x^{a+b} = x^a x^b$.

(b) If $a, b \in \mathbb{Z}$, then $(x^a)^b = x^{ab}$.

Don't forget to handle the case when either a or b is nonpositive.

Problem 5. Assume G is a group and let $a, b \in G$. Is it true that $(ab)^n = a^n b^n$? If not, under what minimal conditions would it be true? Prove the statement that you think is true.

Problem 6. Assume G is a group. Prove that if $x^2 = e$ for all $x \in G$, then G is abelian.

Problem 7. Assume $(G,\star)$ is a group and let H be a nonempty subset of G that is (i) closed under $\star$ and (ii) closed under inverses (i.e., for all $h,k \in H$, (i) $hk \in H$ and (ii) $h^{-1} \in H$). Prove that H is a group under $\star$ in its own right. Such a subset is called a *subgroup*.

Problem 8. Assume G is a group. Prove that if $x \in G$ such that $x^n \neq e$ for all $n \in \mathbb{Z}^+$, then $x^i \neq x^j$ for all $i \neq j$.

Problem 9. Assume $G = \{e,a,b,c\}$ is a group under $\star$ with the property that $x^2 = x^4$ for all $x \in G$ (where e is the identity). Complete the following *group table*, where $x \star y$ is defined to be the entry in the row labeled by x and the column labeled by y.

$\star$	e	a	b	c
e	e	a	b	c
a	a			
b	b			
c	c			

Is your table unique? That is, did you have to fill it out the way you did? Deduce that G is abelian.

Problem 10. (Optional) Assume G is a finite group. Prove that every element of G must appear exactly once in every row and column of the group table for G. (Of course, we are not counting the row and column headings.)