Homework 3

Abstract Algebra I

Complete the following problems. Assume that the dihedral group of order $2n$ is defined by $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$.

Problem 1. Consider the dihedral group D_{10}.

(a) Draw the Cayley diagram for D_{10} using the generators r and s. Be sure to label the vertices as words in r and s and use two different colors for the edges.

(b) Find the order of each of the elements in D_{10}.

Problem 2. Consider the dihedral group D_{2n}.

(a) Use the generators and relations of D_{2n} to show that every element of D_{2n} that is not a power of r has order 2.

(b) Prove that D_{2n} is generated by s and sr.

(c) Prove that $\langle a, b \mid a^2 = b^2 = (ab)^n = 1 \rangle$ gives a presentation for D_{2n} in terms of the generators $a = s$ and $b = sr$. *Hint:* Check that each set of relations can be built from the other.

(d) Draw the Cayley diagram for D_{10} using the generators s and sr. Be sure to label the vertices and use two different colors for the edges.

Problem 3. Consider the dihedral group D_{2n}.

(a) If n is odd and $n \geq 3$, prove that the identity is the only element of D_{2n} that commutes with all the elements of D_{2n}.

(b) What happens if n is even and at least 4? That is, if n is even and $n \geq 4$, are there any non-identity elements in D_{2n} that commute with all of the elements of D_{2n}? You do not need to prove your claim.

Problem 4. Let $Y = \langle u, v \mid u^4 = v^3 = 1, uv = v^2u^2 \rangle$.

(a) Prove that $v^2 = v^{-1}$.

(b) Prove that v commutes with u^3. *Hint:* Try messing with $(v^2u^2)(uv)$.

(c) Prove that v commutes with u. *Hint:* First produce $u^9 = u$ and then use a previous part.

(d) Prove that $uv = 1$.

(e) Prove that $u = 1 = v$ and deduce that $Y = \{1\}$. *Hint:* Do something clever with $u^4v^3 = 1$.

The upshot of this problem is that sometimes innocent-looking presentations can collapse to the trivial group.

Problem 5. Let G be the group of rigid motions in $\mathbb{R}^3$ of a cube. Prove that $|G| = 24$. *Hint:* One approach is to find the number of positions to which an adjacent pair of vertices can be sent.

Problem 6. Consider the symmetric group S_3.

(a) Explicitly show that the adjacent 2-cycles $(1,2)$ and $(2,3)$ generate S_3.

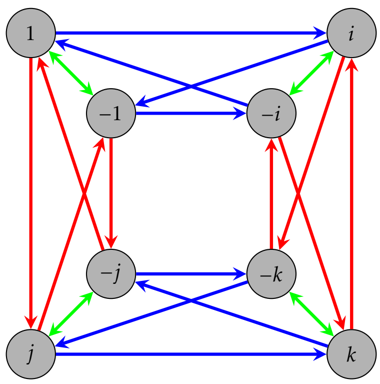
(b) Draw the Cayley diagram for S_3 using $(1,2)$ and $(2,3)$ as generators. Be sure to label the vertices and use two different colors for the edges.

Problem 7. Find the order of $(1,12,8,10,4)(2,13)(5,11,7)(6,9)$ in S_{13}.

Problem 8. Prove that if $\sigma = (a_1, a_2, \ldots, a_m) \in S_n$, then $|\sigma| = m$.

Problem 9. Prove that the order of an element in S_n equals the least common multiple of the lengths of the cycles in its cycle decomposition.

Problem 10. We define the **quaternion group** to be the group $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$ having the Cayley diagram with generators $i, j, -1$ given below. In this case, 1 is the identity.



(a) Create the group table for Q_8.

(b) Find the orders of each of the elements in Q_8.

(c) The Cayley diagram given above was created using a generating set of size 3. Can you identify a smaller generating set?